

# QUASI-PERIODIC SOLUTIONS FOR NONLINEAR WAVE EQUATIONS

W.-M. WANG

ABSTRACT. We construct time quasi-periodic solutions to the nonlinear wave equations on the torus in arbitrary dimensions. This generalizes the method developed in the limit-elliptic setting in [W1] to the hyperbolic setting. The additional ingredient is a Diophantine property of algebraic numbers.

## CONTENTS

1. Introduction and statement of the Theorem
2. The good linear solutions
3. Extraction of parameters
4. The first step
5. The  $\theta$  estimates
6. Proof of the Theorem

## 1. Introduction and statement of the Theorem

We consider *real* valued solutions to the nonlinear wave equation (NLW) on the  $d$ -torus  $\mathbb{T}^d = [0, 2\pi)^d$ :

$$\frac{\partial^2 v}{\partial t^2} - \Delta v + v + v^{p+1} + H(x, v) = 0, \quad (1.1)$$

where  $p \in \mathbb{N}$  and  $p \geq 1$ ; considered as a function on  $\mathbb{R}^d$ ,  $v$  satisfies :  $v(\cdot, x) = v(\cdot, x + 2j\pi)$ ,  $x \in [0, 2\pi)^d$  for all  $j \in \mathbb{Z}^d$ ;  $H(x, v)$  is analytic in  $x$  and  $v$  and has the expansion:

$$H(x, v) = \sum_{m=p+2}^{\infty} \alpha_m(x) v^m,$$

where  $\alpha_m$  as a function on  $\mathbb{R}^d$  is  $(2\pi)^d$  periodic and real and analytic in a strip of width  $\mathcal{O}(1)$  for all  $m$ . The integer  $p$  in (1.1) is *arbitrary*.

We use the standard ordinary differential equation technique to write (1.1) as a first order equation in  $t$ . Let

$$D = \sqrt{-\Delta + 1} \quad (1.2)$$

and

$$u = (v, D^{-1} \frac{\partial v}{\partial t}) \in \mathbb{R}^2. \quad (1.3)$$

Identifying  $\mathbb{R}^2$  with  $\mathbb{C}$ , we then obtain the corresponding first order equation

$$i \frac{\partial u}{\partial t} = Du + D^{-1} [(\frac{u + \bar{u}}{2})^{p+1} + H(x, \frac{u + \bar{u}}{2})]. \quad (1.4)$$

Using Fourier series, the solutions to the linear equation:

$$-i \frac{\partial u}{\partial t} + Du = 0$$

are linear combinations of eigenfunction solutions of the form:

$$e^{-i(\sqrt{j^2+1})t} e^{ij \cdot x}, \quad j \in \mathbb{Z}^d,$$

where  $j^2 = |j|^2$  and  $\cdot$  is the usual inner product. These solutions are either periodic or quasi-periodic in time.

After the addition of the nonlinear terms, it is natural to investigate the bifurcation of these quasi-periodic solutions. We use the space-time approach initiated in [W1], which seems almost necessary here.

## 1.1 The space-time Fourier series.

Let  $u^{(0)}$  be a solution of finite number of frequencies,  $b$  frequencies, to the linear equation:

$$i \frac{\partial u^{(0)}}{\partial t} = D u^{(0)}, \quad (1.5)$$

$$u^{(0)}(t, x) = \sum_{k=1}^b a_k e^{-i(\sqrt{j_k^2+1})t} e^{ij_k \cdot x}.$$

For the nonlinear construction, it is useful to add a dimension for each frequency in time and view  $u^{(0)}$  as a function on  $\mathbb{T}^b \times \mathbb{T}^d = \mathbb{T}^{b+d} \supset \mathbb{T}^d$ . Henceforth  $u^{(0)}$  adopts the form:

$$u^{(0)}(t, x) = \sum_{k=1}^b a_k e^{-i(\sqrt{j_k^2+1})t} e^{ij_k \cdot x}$$

$$:= \sum_{k=1}^b \hat{u}(-e_k, j_k) e^{-i(e_k \cdot \omega^{(0)})t} e^{ij_k \cdot x},$$

where  $e_k = (0, 0, \dots, 1, \dots, 0) \in \mathbb{Z}^b$  is a unit vector, with the only non-zero component in the  $k$ th direction,  $\omega^{(0)} = \{\sqrt{j_k^2+1}\}_{k=1}^b$  ( $j_k \neq 0$ ) and  $\hat{u}(-e_k, j_k) = a_k$ . Therefore  $u^{(0)}$  has Fourier support

$$\text{supp } \hat{u}^{(0)} = \{(-e_k, j_k), k = 1, \dots, b\} \subset \mathbb{Z}^{b+d}, \quad (1.6)$$

where  $j_k \neq j_{k'}$  if  $k \neq k'$ .

For the nonlinear equation (1.1), we seek quasi-periodic solutions with  $b$  frequencies in the form of a space-time Fourier series:

$$u(t, x) = \sum_{(n, j)} \hat{u}(n, j) e^{in \cdot \omega t} e^{ij \cdot x}, \quad (n, j) \in \mathbb{Z}^{b+d}, \quad (1.7)$$

with the frequency  $\omega \in \mathbb{R}^b$  to be determined. This is the well-known amplitude-frequency modulation fundamental to nonlinear equations. We note that the corresponding linear solution  $u^{(0)}$  has *fixed* frequency  $\omega = \omega^{(0)} = \{\sqrt{j_k^2+1}\}_{k=1}^b \in \mathbb{R}^b$ , which are eigenvalues of the operator  $D$  in (1.2).

In the Fourier space  $\mathbb{Z}^{b+d}$ , the support of the solution in the form (1.7) to the linear equation (1.5) and its complex conjugate are by definition, the bi-characteristics  $\mathcal{C}$ ,

$$\mathcal{C} = \mathcal{C}_+ \oplus \mathcal{C}_- \subset \mathbb{Z}^{b+d} \oplus \mathbb{Z}^{b+d} \sim \mathbb{Z}^{b+d} \times \{0, 1\}$$

with

$$\mathcal{C}_{\pm} = \{(n, j) \in \mathbb{Z}^{b+d} \mid \pm n \cdot \omega^{(0)} + \sqrt{j^2+1} = 0\}. \quad (1.8)$$

The set  $\mathcal{C}$  is the support of the solutions to the linear equation (1.5) in the form (1.7) and is the resonant or singular set for the nonlinear equation (1.1). We consider  $\mathcal{C}$  as the restriction to  $\mathbb{Z}^{b+d} \times \{0, 1\}$  of the corresponding manifold, the hyperboloids on  $\mathbb{R}^{b+d} \times \{0, 1\}$ .

We say that a solution to the linear equation (1.5) is *good* if its space-time (spatial-frequency) Fourier support satisfies the non-degeneracy conditions (i-iii) in sect. 2.1. In that case we also say that its real part, which is a real solution to the linear wave equation:

$$\frac{\partial^2 v^{(0)}}{\partial t^2} - \Delta v^{(0)} + v^{(0)} = 0, \quad (1.9)$$

is *good*.

## 1.2 The main result.

Under the assumption that  $H$  is a polynomial in  $u$ ,  $\bar{u}$ ,  $e^{ix_k}$  and  $e^{-ix_k}$ ,  $k = 1, 2, \dots, b$ ,  $x_k \in [0, 2\pi)$ , the main result is:

**Theorem.** *Assume that*

$$v^{(0)}(t, x) = \operatorname{Re} \sum_{k=1}^b a_k e^{-i(\sqrt{j_k^2+1})t} e^{ij_k \cdot x}$$

*is good, satisfying the non-degeneracy conditions (i-iii),  $a = \{a_k\} \in (0, \delta]^b = \mathcal{B}(0, \delta)$  and  $p$  even. Assume that  $b > C_p d$ , where  $C_p$  only depends on  $p$ . Then for all  $0 < \epsilon < 1$ , there exists  $\delta_0 > 0$ , such that for all  $\delta \in (0, \delta_0)$ , there is a Cantor set  $\mathcal{G} \subset \mathcal{B}(0, \delta)$  with*

$$\operatorname{meas} \mathcal{G} / \delta^b \geq 1 - \epsilon. \quad (1.10)$$

*For all  $a \in \mathcal{G}$ , there is a quasi-periodic solution of  $b$  frequencies to the nonlinear wave equation (1.1):*

$$v(t, x) = \operatorname{Re} \sum a_k e^{-i\omega_k t} e^{ij_k \cdot x} + o(\delta^{3/2}),$$

*with basic frequencies  $\omega = \omega(a) = \{\omega_k(a)\}_{k=1}^b$  satisfying*

$$\omega_k = \sqrt{j_k^2 + 1} + \mathcal{O}(\delta^p),$$

*and the amplitude-frequency map  $a \mapsto \omega(a)$  is a diffeomorphism in the sense of Whitney; the remainder  $o(\delta^{3/2})$  is a Gevrey function on  $\mathbb{R} \times \mathbb{T}^d$ .*

*Remark 1.* The above Theorem first appeared in the announcement [W2].

*Remark 2.* The concept of good linear solutions remains valid for odd  $p$ . It is determined entirely by the leading order nonlinear term  $v^{p+1}$ . The additional assumption of

even  $p$  is to ensure amplitude-frequency modulation at the leading order  $\mathcal{O}(\delta^p)$ . This is a sufficient but *not* necessary condition. The parities in  $v$  of the higher order terms do not matter as they are treated as perturbations. The condition of large  $b$ , namely  $b > C_p d$ , is imposed in order that certain determinants are not identically zero, cf. sect. 4. This is the same reason as in [W1]. It cannot be excluded that this condition could be improved after more technical work. Contrary to [W1], however, aside from the non-degeneracy conditions, this is the only other condition needed to prove the Theorem. This is because the forthcoming non-degeneracy condition (ii) in sect. 2.1 dictates that  $\omega^{(0)}$  is Diophantine; moreover the mass term 1, in the wave operator  $D$ , introduces curvature, cf. sect. 2.2.

*Remark 3.* The polynomial restriction on  $H$  is technical; it is in order to use directly the analysis in [B3] to estimate the Green's functions of the linearized operators. In the bifurcation analysis in sects. 2-4, 5.1, this restriction is not needed. The result in the Theorem most likely remains valid for analytic  $H$ . This is as in [W1] for NLS.

*Remark 4.* As  $\delta \rightarrow 0$ ,  $\text{meas } \mathcal{G}/\delta^b \not\rightarrow 1$ , in order that the amplitude-frequency map stays non-degenerate:

$$\|(\frac{\partial \omega}{\partial a})^{-1}\| \lesssim \mathcal{O}_\epsilon(\delta^{p-1}).$$

This precise  $\mathcal{O}(\delta^{p-1})$  bound is in order to ensure the invertibility of appropriate linearized operators, and seems to be an infinite dimensional (phase space) feature, cf. the proofs of Lemmas 4.1 and 5.4. We note that in the non-resonant setting, the corresponding parameter-frequency map is non-degenerate with an upper bound of the order  $\mathcal{O}(1)$ , so the above problem does not arise and as  $\delta \rightarrow 0$ , the Cantor set tends to a set of full measure, cf. [B3, 4].

*Remark 5.* The search for special solutions to the NLW in (1.1) has a long history, dating back from the 1978 paper of Rabinowitz on existence of periodic solutions to (1.1) in one dimension under Dirichlet boundary conditions, cf. [Ra, BCN]. These are, however, different types of solutions with prescribed rational frequencies, constructed using variational methods. Lastly, we note that when  $p \geq \frac{4}{d-2}$  ( $d \geq 3$ ), global solutions to (1.1) do not seem to be known in general.

This Theorem appears to be the first general existence result on quasi-periodic solutions to the NLW in (1.1) in arbitrary dimensions. Previously quasi-periodic solutions only seem to have been constructed in one dimension with positive mass  $m$ . In that case, the linear wave equation:

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} + mv = 0,$$

gives rise to an eigenvalue set  $\{\sqrt{j^2 + m}, j \in \mathbb{Z}\}$  close to the set of integers, see [B2, P] and [CY, K, Way] in a related context. For almost all  $m$ , this set is linearly independent over the integers. This property does not have higher dimensional analogues

and seems to have been a serious obstacle. (In the special time periodic case, solutions have been constructed in higher dimensions in [B1].) In [B4], Chap. 20, however, Bourgain constructed quasi-periodic solutions to a variant of NLW with Fourier multipliers acting as parameters, to avoid the resonances and to make the problem more “elliptic”. The latter aspect is new, due to the infinite dimensional phase space, and specific to hyperbolic partial differential equations such as NLW.

To insert into a more general context, it is known, cf. [G] for example, that the spectrum of an elliptic first-order operator on a generic compact manifold is *dense*. Since the spectrum of the wave operator  $D$  in (1.2) on the flat torus is the set

$$\{\sqrt{j^2 + 1} \mid j \in \mathbb{Z}^d\},$$

which becomes dense in  $d \geq 2$ , it can serve as a model example of the generic case.

*Remark.* We note that “genuine” hyperbolicity occurs only in dimensions two and above.

### 1.3 Hyperbolicity and small-divisors.

The NLW in (1.1) gives rise to an infinite dimensional dynamical system. Studying such a system usually requires certain separation property in order to approximate it by “direct sums” of finite dimensional systems. This is the same idea as for the nonlinear Schrödinger equations (NLS) in [W1]. Since, as in [W1], there are no external parameters to control the resonances a priori, the bounds on the sizes of the finite dimensional systems need to be optimal.

The bi-characteristics in (1.8) define hyperboloids; while for NLS the bi-characteristics are paraboloids, which are limit-elliptic, cf. (1.5, 1.6) in sect 1.1 of [W1]. Due to the convexity or ellipticity of the Laplacian, NLS essentially corresponds to elliptic geometry. Separation property is attained by considering intersections of hyperplanes and occasionally ellipsoids, which are compact [W1].

For NLW, the geometry is *hyperbolic*, separation entails intersections of quadratic hyper-surfaces, which are generically non-compact, cf. sects. 5.1 and 2. It is much more difficult to discern that a large system of quadratic polynomial equations in many variables has no solution. To circumvent this difficulty, a pre-selection of the initial frequencies  $\omega^{(0)} = \{\omega_k^{(0)}\}_{k=1}^b$  is made along number theoretical considerations. More precisely, the frequencies are chosen to be square roots of distinct *square free* integers. This is the key new feature, compared to NLS.

As a direct consequence of this frequency selection, there is the linear independence, cf. e.g., [Ro] for a proof (for notational simplicity,  $\omega$  is generally written for  $\omega^{(0)}$ ) :

$$\|n \cdot \omega\|_{\mathbb{T}} \neq 0,$$

where  $n \neq 0$  and  $\|\cdot\|_{\mathbb{T}}$  denotes the distance to the integers; as well as the *quadratic* non-equality:

$$\left\| \sum_{k < \ell} n_k n_\ell \omega_k \omega_\ell \right\|_{\mathbb{T}} \neq 0,$$

where  $\sum_{k < \ell} |n_k n_\ell| \neq 0$ . (See the elementary derivation in sect. 2.1.) The latter implies that:

$$\left\| \sum_{k, \ell} n_k n_\ell \omega_k \omega_\ell \right\|_{\mathbb{T}} = 0 \iff \sum_{k, \ell} n_k n_\ell \omega_k \omega_\ell = n_k^2 \omega_k^2, \quad (1.11)$$

for some  $k \in \{1, 2, \dots, b\}$ .

The linear independence is the usual one. The quadratic non-equality is new and takes care of hyperbolicity; moreover it *doubles* as a small-divisor lower bound for the dense linear flow: If

$$\pm n \cdot \omega + \sqrt{j^2 + 1} \neq 0, \quad (1.12)$$

then

$$|\pm n \cdot \omega + \sqrt{j^2 + 1}| > c|n|^{-\alpha}, \quad n \neq 0, \quad (1.13)$$

where  $c, \alpha > 0$ , by using the simultaneous Diophantine approximation result in [Schm], cf. the very beginning of sect. 4.1. We note that these are new types of small-divisors, which do not appear in NLS.

The expressions on the left side of (1.12) appear as the principal symbol of an appropriate linearized operator, which is the “divisor” in the problem. They define the bi-characteristics in (1.8) when set equal to 0. These “zero-divisors” are dealt with as in [W1], which essentially uses the sub-principal symbol of the linearized operator (the linearization of the nonlinear term, cf. (2.5) below) to control the small eigenvalues under certain non-degeneracy conditions, the to be stated conditions (i-iii). Combining with the small-divisor estimates in (1.12), enables amplitude-frequency modulation. Afterwards, one proceeds similarly to [W1] to complete the proof of the Theorem.

The interdependence of hyperbolicity, number theory and small-divisors revealed by NLW seems new. In [B4], Chap. 20, the frequencies are the Fourier multipliers  $\omega$ . Nonlinear Diophantine conditions are imposed on  $\omega$  to achieve a certain degree of “ellipticity” – in fact, these conditions coincide with the corresponding bi-characteristics hyperboloids, defined as in (1.8), being an *empty* set – in order to obtain a separation property nearby, cf. Lemma 20.14 in Chap. 20, [B4]. (Recall that for NLS, this is *not* needed, as the bi-characteristics are paraboloids and are *limit-elliptic* – there is an a priori separation property, cf. Lemma 19.10 in Chap. 19, [B4].) Since  $\omega$  is a parameter, this amounts to estimate the measure of the set in  $\omega$  on which the nonlinear polynomials are small.

For the NLW in (1.1), however, the frequency vector  $\omega^{(0)}$  is *fixed* – a fixed algebraic vector – some of these nonlinear conditions – namely the ones produced by (1.11)

– simply cannot be imposed – the bi-characteristics defined in (1.8) are *always non-empty and hyperbolic* – the separation Lemma 20.14 in [B4] cannot be validated. It is essentially necessary to rely on the sub-principal symbol to attain any separation property, since it cannot come from the hyperbolic principal symbol. This is “indigenously” infinite dimensional and *the* novelty of the paper. Consequently, the approach here is significantly different from [B4] and appeals to number theory.

*Remark.* We note that in the case of NLS, in sects. 2 and 3 of [W1], separation property was also derived using the sub-principal symbol. It gives quantitative and more precise bounds on the sizes of “connected sets” (the finite dimensional systems) on the bi-characteristics than that in Lemma 19.10 in [B4], which is proven using the ellipticity of the principal symbol. These precise bounds permit amplitude-frequency modulation, as indicated earlier.

#### 1.4 Effective resultant analysis.

As initiated in [W1], to realize the aforementioned finite dimensional approximations, we describe the geometry algebraically and make variable reductions. This leads to the concept of *good* linear solutions  $u^{(0)}$ . The “effective resultant” analysis introduced in [W1] is used to control the sizes of the finite dimensional systems. There is, however, a fundamental difference between NLS and NLW.

For NLS, the bi-characteristics polynomials are *quadratic* in the spatial Fourier variable  $j$ , reflecting the presence of the Laplacian, cf. (1.5, 1.6) in sect 1.1 of [W1]; while for NLW, it is asymptotically *linear* at infinity, due to the wave operator.

For NLS, the effective resultants  $\mathcal{D}$  can be reached with relative ease – since as mentioned earlier – it mostly involves intersections of hyper-planes. Subsequently, the condition  $\mathcal{D} \neq 0$  may be imposed, after using (what can be viewed as) an algebraic version of a variational argument, cf. the proof of Lemma 2.5 in sect. 2.2 of [W1].

For NLW this is not feasible – as it involves intersections of hyperboloids – instead – properties of square roots of square free integers are used to arrive at the effective resultants  $\mathcal{D}$ . The variety defined by  $\mathcal{D} = 0$  is then analyzed with the additional help of the curvature near the origin, introduced by the mass term 1 in the wave operator. (For more details, see the proofs of Proposition 2.1 and Lemma 5.2 in sects. 2.2 and 5.1.) The variable reductions here are essentially more difficult than that for NLS.

In its essence, this new bifurcation analysis introduced in [W1] and generalized here is *non-perturbative* – this is what permits the method to surmount the challenges of dense degenerate spectra of differential operators – by incorporating different elements of mathematics.

#### 1.5 A comparison of the proofs for NLW and NLS.

The proof of the present Theorem follows the general approach introduced in [W1] for NLS. However there are substantial differences due to the fact that the eigenvalue



spacing for the wave operator  $D$  tends to zero. As mentioned earlier, this makes the variable reductions much more difficult. Concretely, the NLS proof in [W1] does not use any number theory aside from the spacing of integers being 1; while for NLW, there is the essential square free condition, cf. the non-degeneracy condition (ii) in sect. 2.1. Without (ii), it would be very difficult to imagine a corresponding decomposition into a direct sum of finite dimensional systems as in Proposition 2.1 and Lemma 5.2. We further note that Lemma 5.2 generalizes Proposition 2.1 to bi-characteristics

$$\mathcal{C}_{\pm} = \{(n, j) | \pm n \cdot \omega + \sqrt{j^2 + 1} = \Theta\}$$

for  $\Theta \in \mathbb{R}$  (in (1.8),  $\Theta = 0$ ), i. e., to higher part of the spectrum – essential for the Newton iteration.

For NLS, the proofs of the two corresponding decompositions in [W1] are essentially the same, since the eigenvalue spacing remains 1 in the higher part of the spectrum; while for NLW the proof of Lemma 5.2 is very different from that of Proposition 2.1, much more difficult and relying in an essential way on the square free condition (ii). The other, related difference is that NLS only needs the usual (linear) Diophantine condition; while NLW, also the quadratic non-equalities in (††) and later (5.9, 5.10).

Once the decomposition (compactification) is achieved as in Proposition 2.1 for the lower part of the spectrum and Lemma 5.2 for the higher part, the analysis of the Newton scheme proceeds similarly to that for NLS, both based on the blueprint in Chaps. 19 and 20 of [B4].

## 1.6 Layout of the paper.

Using the ansatz in (1.7), the NLW in (1.1) is transformed into a nonlinear matrix equation in the Fourier coefficients  $\hat{u}(n, j)$ . (See (2.2), sects. 3 and 4 for more details.) The equation is divided into two parts using a Lyapunov-Schmidt  $P$  and  $Q$  equations decomposition. The domain of the  $Q$ -equations is the set  $\text{supp } \hat{u}^{(0)}$  defined in (1.6); while that of the  $P$ -equations, the complement set. This is as initiated in [W1]. The  $Q$ -equations are used to solve for the frequencies  $\omega$ ; while the  $P$ -equations, the Fourier coefficients  $\hat{u}$ , iteratively using a multi-scale Newton scheme. As in [W1], due to the resonances, the equation is linearized at the unperturbed solution  $u^{(0)}$ , instead of at 0. Ensuring invertibility of the linearized operator then leads to the non-degeneracy conditions on  $u^{(0)}$ .

More specifically, in sect. 2, good linear solutions and connected sets are defined. The non-degeneracy conditions are then used to bound the sizes of connected sets on the bi-characteristics  $\mathcal{C}$  – the sizes of the “finite dimensional” systems. The  $Q$ -equations are solved in sect. 3, leading to amplitude-frequency modulation. In sect. 4, the first corrections to the linear solutions are obtained using the Newton scheme. The preparations for all subsequent Newton iterations are done in sect. 5. Subsequently, sect. 6 completes the proof.

## Notations

We summarize below some of the notational conventions:

- The dimension  $d$ , the degree of nonlinearity  $p$  and the number of basic frequencies  $b$  are fixed. The set  $\{j_k\}_{k=1}^b$  is a fixed subset of  $\mathbb{Z}^d$ .
- The letter  $u$  denotes a function on  $\mathbb{T}^{b+d}$ ,  $\hat{u}$  its Fourier series. The hat is generally dropped and  $u$  is written for  $\hat{u}$ , which are functions on  $\mathbb{Z}^{b+d}$ .
- The letters  $n$  and  $\nu$  denote vectors in  $\mathbb{Z}^b$ ; while  $j$  and  $\eta$  vectors in  $\mathbb{Z}^d$ .
- The dot  $\cdot$  denotes the usual scalar product in Euclidean space. To simplify notations, one writes  $j^2$  for  $j \cdot j$  etc.
- The norm  $\|\cdot\|$  stands for the  $\ell^2$  or operator norm; while  $||$  for the sup-norm or the length of a vector in a finite dimensional vector space or the number of elements in a given set.
- An identically zero function  $f$  is denoted by  $f \equiv 0$ ; the negation  $f \neq 0$ .
- Given two positive quantities,  $A, B > 0$ ,  $A \asymp B$  signifies  $cB < A < CB$  for some  $0 < c < C$ ; while  $A \lesssim B$ ,  $A < CB$ .
- Large positive constants are generally denoted by upper case letters such as  $C, C'$  etc.; while small ones  $c, c', \epsilon$  etc. Unless indicated otherwise, they are not the same and may vary from statement to statement.

## 2. The good linear solutions

Using the ansatz (1.7), (1.4) becomes

$$\text{diag } (n \cdot \omega + \sqrt{j^2 + 1}) \hat{u} + \text{diag } (1/\sqrt{j^2 + 1}) [(\frac{\hat{u} + \hat{\bar{u}}}{2})^{*(p+1)} + \sum_{m=2}^{\infty} \hat{\alpha}_m * (\frac{\hat{u} + \hat{\bar{u}}}{2})^{*(p+m)}] = 0,$$

which is equivalent to

$$\text{diag } [\sqrt{j^2 + 1}(n \cdot \omega + \sqrt{j^2 + 1})] \hat{u} + (\frac{\hat{u} + \hat{\bar{u}}}{2})^{*(p+1)} + \sum_{m=2}^{\infty} \hat{\alpha}_m * (\frac{\hat{u} + \hat{\bar{u}}}{2})^{*(p+m)} = 0, \quad (2.1)$$

where  $(n, j) \in \mathbb{Z}^{b+d}$ ,  $\omega \in \mathbb{R}^b$  is to be determined and

$$|\hat{\alpha}_m(\ell)| \leq C e^{-c|\ell|} \quad (C, c > 0)$$

for all  $m$ . From now on we work with (2.1), for simplicity we drop the hat and write  $u$  for  $\hat{u}$  and  $\bar{u}$  for  $\hat{\bar{u}}$  etc.

We complete (2.1) by writing the equation for the complex conjugate. So we have

$$\begin{cases} \text{diag} [\sqrt{j^2 + 1}(n \cdot \omega + \sqrt{j^2 + 1})]u + (\frac{u+\bar{u}}{2})^{*(p+1)} + \sum_{m=2}^{\infty} \alpha_m * (\frac{u+\bar{u}}{2})^{*(p+m)} = 0, \\ \text{diag} [\sqrt{j^2 + 1}(-n \cdot \omega + \sqrt{j^2 + 1})]\bar{u} + (\frac{u+\bar{u}}{2})^{*(p+1)} + \sum_{m=2}^{\infty} \bar{\alpha}_m * (\frac{u+\bar{u}}{2})^{*(p+m)} = 0. \end{cases} \quad (2.2)$$

We seek solutions close to the linear solution  $u^{(0)}$  of  $b$  frequencies,  $\text{supp } u^{(0)} = \{(-e_k, j_k), k = 1, \dots, b\}$ , with frequencies  $\omega^{(0)} = \{\sqrt{j_k^2 + 1}\}_{k=1}^b$  ( $j_k \neq 0$ ) and small amplitudes  $a = \{a_k\}_{k=1}^b$  satisfying  $\|a\| = \mathcal{O}(\delta) \ll 1$ . Denote the left side of (2.2) by  $F(u, \bar{u})$ .

Linearizing at  $(u^{(0)}, \bar{u}^{(0)})$ , we are led to study the linearized operator  $F'(u^{(0)}, \bar{u}^{(0)})$  on  $\ell^2(\mathbb{Z}^{b+d}) \times \ell^2(\mathbb{Z}^{b+d})$  with

$$F' = D' + A + \mathcal{O}(\delta^{p+1}), \quad (2.3)$$

where

$$D' = \begin{pmatrix} \text{diag} [\sqrt{j^2 + 1}(n \cdot \omega + \sqrt{j^2 + 1})] & 0 \\ 0 & \text{diag} [\sqrt{j^2 + 1}(-n \cdot \omega + \sqrt{j^2 + 1})] \end{pmatrix} \quad (2.4)$$

and

$$A = \frac{p+1}{2^{p+1}} \begin{pmatrix} (u^{(0)} + \bar{u}^{(0)})^{*p} & (u^{(0)} + \bar{u}^{(0)})^{*p} \\ (u^{(0)} + \bar{u}^{(0)})^{*p} & (u^{(0)} + \bar{u}^{(0)})^{*p} \end{pmatrix}. \quad (2.5)$$

In order for  $u^{(0)}$  to bifurcate to a nearby solution  $u$  to the nonlinear equation in (1.1), algebro-geometric conditions will be imposed on the Fourier support of  $u^{(0)}$ . This is in the spirit of [W1]. The algebraic aspect here is, however, new, and originates from the number theoretical considerations for the frequencies (support) of  $u^{(0)}$ .

## 2.1 The good linear solutions.

To define good  $u^{(0)}$ , we need to analyze the convolution matrix  $A$  defined in (2.5). Let

$$\Gamma = \text{supp} [(u^{(0)} + \bar{u}^{(0)})^{*p}] \setminus \{(0, 0)\} = \{(\nu, \eta)\} \subset \mathbb{Z}^{b+d}, \quad (2.6)$$

with

$$\text{supp } u^{(0)} = \{(-e_k, j_k)\}_{k=1}^b.$$

From the definition,  $(\nu, \eta) \in \Gamma$  are of the form

$$(-\sum_{k=1}^b m_k e_k, \sum_{k=1}^b m_k j_k),$$

11

where  $\sum_{k=1}^b |m_k| \leq p$ . So

$$\eta = - \sum_{i=1}^b \nu_i j_i.$$

More generally, for any fixed  $R \in \mathbb{N}$ , consider the set

$$\tilde{\Gamma} = \bigcup_{r=1}^R \text{supp} [(u^{(0)} + \bar{u}^{(0)})^{*pr}] \setminus \{(0, 0)\}, \quad (\#)$$

and again use  $(\nu, \eta)$  to denote an element of  $\tilde{\Gamma}$ :  $(\nu, \eta) \in \tilde{\Gamma}$ . If  $(\nu, \eta) \in \tilde{\Gamma}$ , then  $(\nu, \eta)$  is of the form

$$(- \sum_{k=1}^b m_k e_k, \sum_{k=1}^b m_k j_k),$$

where  $\sum_{k=1}^b |m_k| \leq pR$ . So there remains the relation:

$$\eta = - \sum_{i=1}^b \nu_i j_i, \quad (\#\#)$$

for  $(\nu, \eta) \in \tilde{\Gamma}$ .

As in [W1],  $\eta = \eta(j_1, j_2, \dots, j_b)$  is considered as a function from  $(\mathbb{Z}^d)^b$  to  $\mathbb{Z}^d$ . More precisely, for a given  $\nu = \{\nu_i\}_{i=1}^b \in \mathbb{Z}^b$ ,  $\eta = - \sum_{i=1}^b \nu_i j_i$  is a function from  $(\mathbb{Z}^d)^b$  to  $\mathbb{Z}^d$ . Recall also from [W1] the fundamental relation:

$$\nu = 0 \iff \eta \equiv 0. \quad (\star)$$

(See sect. 2 of [W1], for more details on these basic points.)

For the wave equation,  $\eta$  which is a function of at most 2 variables,  $\eta = \eta(j_k, j_\ell)$ ,  $k, \ell \in \{1, 2, \dots, b\}$ , plays an important role. Below  $(\nu, \eta)$  is again considered as a point in  $\mathbb{Z}^{b+d}$ . Fix  $R = 2d + 1$ .

**Definition.**  $u^{(0)}$  a solution of  $b$  frequencies  $j_1, j_2, \dots, j_b$ , to the linear equation in (1.5) is *good* if the following three conditions are satisfied:

- (i) Let  $\bar{d} = \min(d, b)$ . Any  $\bar{d}$  vectors in the set  $\{j_k\}_{k=1}^b$  are linearly independent. For all  $j_k$ ,  $k = 1, 2, \dots, b$ , define the set of differences

$$J_k = \{j_{k'} - j_k | k' = 1, \dots, b, k' \neq k\}.$$

If  $b \geq d + 1$ , any  $d$  vectors in  $J_k$  are linearly independent. (If  $b \leq d$ , there is no condition (i).)

(ii) The integers  $(j_k^2 + 1)$ ,  $k = 1, 2, \dots, b$ , are distinct:

$$1 < j_1^2 + 1 < j_2^2 + 1 < \dots < j_b^2 + 1,$$

and square free.

(iii) For all given  $k \in \{1, 2, \dots, b\}$  and  $m \in \mathbb{Z} \cap [-p, p] \setminus \{0\}$ , consider the set of  $(\nu, \eta) \in \tilde{\Gamma}$  with

$$\nu = -me_k + m_\ell e_\ell,$$

where  $\ell \in \{1, 2, \dots, b\}$ ,  $m_\ell \in \mathbb{Z}$ ,  $|m_\ell| \leq 2pd$ , and

$$\begin{aligned} \eta &= mj_k - m_\ell j_\ell \\ &:= \eta(\ell, m_\ell) \neq 0. \end{aligned}$$

For each  $\eta$ , define  $L$  to be

$$L = 2m\eta \cdot j_k + (m^2 - m_\ell^2) := L(\ell, m_\ell).$$

Denote by  $P(\ell, m_\ell)$  the corresponding  $d$ -dimensional hyperplane in  $\mathbb{R}^d$ :

$$2\eta \cdot j + L = 0, \tag{*}$$

where  $\eta = \eta(\ell, m_\ell)$  and  $L = L(\ell, m_\ell)$ .

Let  $\sigma$  be any set of  $(\ell, m_\ell)$  with  $2d$  elements,  $\ell \in \{1, 2, \dots, b\}$ ,  $m_\ell \in \mathbb{Z}$ ,  $|m_\ell| \leq 2pd$ , such that there exists  $(\tilde{\ell}, m_{\tilde{\ell}}) \in \sigma$  with  $m_{\tilde{\ell}} \neq \pm m$ , then

$$\bigcap_{\sigma} P(\ell, m_\ell) = \emptyset.$$

*Remark.* In lieu of condition (ii),  $(j_k^2 + 1)$  may be taken to be multiples of *distinct* square free integers – similarly, instead of 1, the mass may be fixed at any rational number – the proof of the Theorem is the same. We also note that contrary to (ii), (i, iii) do not use the integer nature of  $j_k$ ,  $k = 1, 2, \dots, b$ . So the hyperplanes in (\*) are indeed considered to be in  $\mathbb{R}^d$ .

As mentioned in sect. 1, condition (ii) implies the usual linear independence:

$$\left\| \sum_{k=1}^b n_k \omega_k \right\|_{\mathbb{T}} \neq 0, \tag{\dagger}$$

where  $n_k \in \mathbb{Z}$ ,  $\sum_k |n_k| \neq 0$ ; as well as the quadratic non-equality:

$$\left\| \sum_{k, \ell; k < \ell} n_k n_\ell \omega_k \omega_\ell \right\|_{\mathbb{T}} \neq 0, \quad (\dagger\dagger)$$

where  $\sum_{k < \ell} |n_k n_\ell| \neq 0$ .

Property  $(\dagger)$  follows from basic algebra ([Ro]). To prove  $(\dagger\dagger)$ , we notice that condition (ii) indicates that

$$\sum_{k=1}^b n_k \omega_k \pm \sqrt{N} \neq 0,$$

for all  $N \in \mathbb{N}$ , if there are  $n_\ell \neq 0$ ,  $n_{\ell'} \neq 0$  for  $\ell \neq \ell'$ . This can be seen as follows. Write

$$\sqrt{N} = q\sqrt{N'},$$

where  $q \in \mathbb{N}$  and  $N'$  is square free. If

$$\sum_{k=1}^b n_k \omega_k \pm \sqrt{N} = \sum_{k=1}^b n_k \omega_k \pm q\sqrt{N'} = 0,$$

then there must be  $\omega_i$  such that  $\sqrt{N'} = \omega_i$ . So

$$\sum_{k \neq i} n_k \omega_k + (n_i \pm q)\omega_i = 0.$$

Now since the first sum contains at least one non-zero  $n_k$  and  $\omega_k \neq \omega_i$ , this contradicts  $(\dagger)$ . Multiplying the two expressions in  $\pm\sqrt{N}$ , and since  $N \in \mathbb{N}$  is arbitrary, yields  $(\dagger\dagger)$ .

The following indicates that the above three conditions are viable.

**Lemma.** *There is an infinite number of  $(j_1, j_2, \dots, j_b) \in (\mathbb{Z}^d)^b$  which satisfy the non-degeneracy conditions (i-iii).*

*Proof.* The first part of condition (i) is satisfied if the  $\bar{d} \times \bar{d}$  determinant of any  $\bar{d}$  vectors in the set  $\{j_k\}_{k=1}^b$  are non-zero. The second part is satisfied similarly. The intersection of these sets  $\mathcal{D}'$  on which (i) holds is Zarisky open and therefore contains an infinite number of integers.

For (iii), since  $(0, 0) \notin \tilde{\Gamma}$ ,  $\eta \neq 0$  define sets of co-dimension 1. Fixing  $m, k$ , we note that if  $\ell = k$  (if  $m_\ell = 0$ , set  $\ell = k$ ), then

$$P(k, m_k) \cap P(k, m'_k) = \emptyset, \quad (2.7)$$

if  $m_k \neq m'_k$ . This is because  $P(k, m_k)$  is the plane defined by the equation:

$$2\eta(k, m_k) \cdot j + L(k, m_k) = 0,$$

equivalently by

$$2j_k \cdot j + 2mj_k^2 + (m + m_k) = 0,$$

which is incompatible with the equation for  $P(k, m'_k)$  if  $m_k \neq m'_k$ . Similarly, when  $\ell \neq k$ ,

$$P(\ell, m_\ell) \cap P(\ell, m'_\ell) \cap P(\ell, m''_\ell) = \emptyset, \quad (2.8)$$

if  $m_\ell \neq m'_\ell$  or  $m_\ell \neq m''_\ell$ .

Below we may assume  $b \geq d$ , as otherwise (2.7, 2.8) imply

$$\bigcap_{\sigma} P(\ell, m_\ell) = \emptyset.$$

Let  $\sigma'$  be a subset of  $\sigma$  with  $d$  elements:

$$\sigma' \subset \sigma \setminus \{(\tilde{\ell}, m_{\tilde{\ell}})\}, |\sigma'| = d.$$

The corresponding set of  $\eta$  is then:

$$\{\eta(\ell, m_\ell) = mj_k - m_\ell j_\ell; (\ell, m_\ell) \in \sigma'\}.$$

One may assume that the above set of  $\eta$  depends on at least  $d$  variables in  $\{j_\kappa\}_{\kappa=1}^b$ , as otherwise if  $\sigma$  does not contain such a subset, then

$$\bigcap_{\sigma \setminus \{(\tilde{\ell}, m_{\tilde{\ell}})\}} P(\ell, m_\ell) = \emptyset$$

from (2.7, 2.8).

Let

$$\tilde{\sigma} = \sigma' \cup \{(\tilde{\ell}, m_{\tilde{\ell}})\}.$$

One stipulates that the  $(d+1) \times (d+1)$  determinant of the set of vectors

$$\begin{aligned} &\{(2\eta(\ell, m_\ell), L(\ell, m_\ell)); (\ell, m_\ell) \in \tilde{\sigma}\}, \\ &\det(2\eta, L) \neq 0. \end{aligned} \quad (2.9)$$

There are two cases.

If  $\tilde{\ell} = k$  or if  $m_{\tilde{\ell}} = 0$ , set  $j_k = 0$ . The first part of condition (i) gives the linear independence of

$$\{\eta(\ell, m_\ell); (\ell, m_\ell) \in \sigma'\},$$

where (2.7) is also used. Since  $L(\tilde{\ell}, m_{\tilde{\ell}}) \neq 0$ , this yields

$$\det(2\eta, L) \neq 0.$$

If  $\tilde{\ell} \neq k$ , set  $\eta_{\tilde{\ell}} = 0$ . The conclusion in (2.9) follows using  $L(\tilde{\ell}, m_{\tilde{\ell}}) \neq 0$  and the linear independence of either

$$\{\eta(\ell, m_{\ell}); (\ell, m_{\ell}) \in \sigma'\},$$

as vectors (not just as functions) or that of

$$\{\eta(\ell, m_{\ell}); (\ell, m_{\ell}) \in \sigma'\}$$

at  $j_k = 0$ . So

$$\bar{D} = \det(2\eta, L) \neq 0.$$

Let  $D(k, m, \tilde{\sigma})$  be the set in  $(\mathbb{R}^d)^b$  defined by  $\bar{D} \neq 0$ . Then  $D(k, m, \tilde{\sigma})$  is Zarisky open containing an infinite number of integers.

More generally, for any fixed  $k, m$ , let  $\tilde{\sigma}$  be a set of  $(d+1)$  elements  $(\ell_i, m_{\ell_i})$ ,  $i = 1, 2, \dots, d+1$ , such that the corresponding subset of  $\eta$  of  $d$  elements:

$$\eta(\ell_i, m_{\ell_i}) = m j_k - m_{\ell_i} j_{\ell_i},$$

$i = 1, 2, \dots, d$ , depends on at least  $d$  variables in  $\{j_{\kappa}\}_{\kappa=1}^b$  and such that  $m_{\ell_{d+1}} \neq \pm m$ . Then the  $(d+1) \times (d+1)$  determinant

$$\bar{D} = \det(2\eta, L) \neq 0$$

by using the same argument. Let  $D(k, m, \tilde{\sigma})$  be the set in  $(\mathbb{R}^d)^b$  defined by  $\bar{D} \neq 0$ . Then the set

$$\mathcal{D} := \bigcap_{k, m, \tilde{\sigma}} D(k, m, \tilde{\sigma}) \neq \emptyset,$$

is Zarisky open and contains an infinite number of integers. On  $\mathcal{D}$ , (iii) is verified.

Call an integer vector  $V$  in  $\mathbb{Z}^d$  square free if all its components are square free. We are left to show that there is an infinite number of

$$(j_1, j_2, \dots, j_b) \in (\mathbb{Z}^d)^b \cap \mathcal{D}' \cap \mathcal{D}, \quad (\diamond)$$

satisfying

$$1 < j_1^2 + 1 < j_2^2 + 1 < \dots < j_b^2 + 1, \quad (\diamond\diamond)$$

and such that the integer vector

$$V = (j_1^2 + 1, j_2^2 + 1, \dots, j_b^2 + 1)$$



is square free. Below we describe such a selection process.

The set  $\mathcal{D}'$  can be described by polynomial non-equalities of degrees at most  $d$ ; while  $\mathcal{D}$  of degrees at most  $d + 2$ . These polynomials are in  $bd$  variables:

$$j_{1,1}, j_{1,2}, \dots, j_{1,d}, \dots, j_{k,i}, \dots, j_{b,d}; k = 1, 2, \dots, b, i = 1, 2, \dots, d,$$

where  $j_{k,i}$  denotes the  $i$ th component of  $j_k \in \mathbb{Z}^d \subset \mathbb{R}^d$ . There are *finite* number (depending only on  $d, p, b$ ),  $N(d, p, b)$  of such polynomials. We shall *first* select the first  $d - 1$  components of each  $j_k$ , in the order  $k = 1, 2, \dots, b$  and then successively the  $d$ th component of  $j_k$  (in the order  $k = 1, 2, \dots, b$ ), i.e., in the following order:

$$j_{1,1}, j_{1,2}, \dots, j_{1,d-1}, \dots, j_{b,1}, \dots, j_{b,d-1}; j_{1,d}, \dots, j_{b,d}.$$

Assume that the variable  $j_{1,1}$  appear in  $N_{1,1}$  polynomials of degrees at most  $d + 2$ . Let  $P$  be such a polynomial. Rename  $j_{1,1}$ ,  $x$  and the other variables collectively  $y$ . Fix  $x = x_0$ . Then  $P(y; x = x_0)$  is a polynomial in  $y$  and there can be at most  $d + 2$  *distinct*  $x_0$ , such that

$$P(y; x = x_0) \equiv 0$$

as a function of  $y$ . This can be seen as follows.

Since  $P \not\equiv 0$ , as a function in  $x, y$ , there exist  $y_0$ , such that

$$P(x; y_0) \not\equiv 0$$

as a function of  $x$ . If there exist  $d + 3$  distinct  $x_0$  such that  $P(x_0, y) \equiv 0$ , then  $P(x_0; y_0) = 0$  for  $d + 3$  distinct  $x_0$ , i.e.,

$$\begin{aligned} P(x_0^{(1)}; y_0) &= 0 \\ P(x_0^{(2)}; y_0) &= 0 \\ &\vdots \\ P(x_0^{(d+2)}; y_0) &= 0. \end{aligned}$$

This contradicts with  $P(x; y_0)$  being a non-zero polynomial in  $x$  of degree at most  $d + 2$ .

Let  $X_{1,1}$  be the set such that if  $x \in X_{1,1}$ , then one of the  $N_{1,1}$  polynomials is identically 0, then the number of elements in  $X_{1,1}$ :

$$|X_{1,1}| \leq N_{1,1}(d + 2) \leq N(d, p, b)(d + 2).$$

(1,1) Fix

$$j_{1,1} \in \mathbb{Z} \setminus X_{1,1} := U_{1,1}.$$

(So the variable  $j_{1,1}$  has been, so to speak, “eliminated”.)

Now consider  $j_{1,2}$ . Using the same reasoning leads to the set of possible zeroes  $X_{1,2}$ , satisfying

$$|X_{1,2}| \leq N_{1,2}(d+2) \leq N(d, p, b)(d+2).$$

(Note that  $X_{1,2}$  depends, of course, on  $j_{1,1}$ , but the upper bound on the number of elements in it does not.)

(1,2) Fix

$$j_{1,2} \in \mathbb{Z} \setminus X_{1,2} := U_{1,2}.$$

$\vdots$

(1,d-1) Fix

$$j_{1,d-1} \in \mathbb{Z} \setminus X_{1,d-1} := U_{1,d-1}.$$

$\vdots$

(b,1) Fix

$$j_{b,1} \in \mathbb{Z} \setminus X_{b,1} := U_{b,1}.$$

$\vdots$

(b,d-1) Fix

$$j_{b,d-1} \in \mathbb{Z} \setminus X_{b,d-1} := U_{b,d-1}.$$

We now only need to select the  $d$ th component of each  $j_k$ ,  $k = 1, 2, \dots, b$ , so that  $V$  satisfies  $(\diamond\diamond)$  and square free.

(1,d) Choose

$$j_{1,d} \in \mathbb{Z} \setminus X_{1,d} := U_{1,d-1},$$

such that

$$j_1^2 + 1 = j_{1,1}^2 + j_{1,2}^2 + \dots + j_{1,d}^2 + 1$$

is square free. This is clearly possible since

$$|X_{1,d}| \leq N_{1,d}(d+2) \leq N(d, p, b)(d+2),$$

and there are infinite number of square free integers. In fact given  $N \in \mathbb{N}$ , the number of square free integers less than or equal to  $N$  is asymptotically,

$$= \frac{6}{\pi^2} N + \mathcal{O}(\sqrt{N}).$$

(2,d) Choose

$$j_{2,d} \in \mathbb{Z} \setminus X_{2,d} := U_{2,d-1},$$

such that

$$j_2^2 + 1 = j_{2,1}^2 + j_{2,2}^2 + \dots + j_{2,d}^2 + 1 > j_1^2 + 1$$

is square free. This is again possible due to finiteness of the set  $X_{2,d}$  and the infiniteness of the set of square free integers.

$\vdots$

(b,d) Choose

$$j_{b,d} \in \mathbb{Z} \setminus X_{b,d} := U_{b,d-1},$$

such that

$$j_b^2 + 1 > j_{b-1}^2 + 1$$

and square free. This shows that there is an infinite number of  $(j_1, j_2, \dots, j_b)$  satisfying  $(\diamond, \diamond\diamond)$ . Hence (i, ii, iii) is satisfied for an infinite number of  $(j_1, j_2, \dots, j_b) \in (\mathbb{Z}^d)^b$  and completes the proof.  $\square$

*Remark.* The second part of the non-degeneracy condition (i) will only be used below to deal with the exceptional case when condition (iii) is not applicable.

## 2.2 Size of connected sets on the bi-characteristics.

Let  $\pi$  be the projection of  $\mathbb{Z}^{b+d} \times \{0, 1\}$  onto  $\mathbb{Z}^{b+d}$ . A set

$$S \subseteq \mathbb{Z}^{b+d} \times \{0, 1\}$$

is called *connected*, if for all  $a, b \in S$ , there exist  $a_1, a_2, \dots, a_m \in S$ , such that

$$\pi a_{k+1} - \pi a_k \in \Gamma, \tag{2.10}$$

for all  $k \in \{0, 1, \dots, m\}$  with  $a_0 := a, a_{m+1} := b$ , where  $\Gamma$  is as defined in (2.6). The number of elements in  $S$ ,  $|S|$ , is its size.

It follows that if  $S$  is connected, then

$$\pi a_k - \pi a_{k'} \in \tilde{\Gamma},$$

for all  $a_k, a_{k'} \in S, k \neq k'$ , where  $\tilde{\Gamma}$  as defined in  $(\#)$  for some  $R > 0$ . Choose an (arbitrary) element  $a_0 \in S$  and call it the root. Then

(P1)  $\pi a_k - \pi a_0 \in \tilde{\Gamma}$ , for all  $k \neq 0$

and there must exist  $k' \neq 0$ , such that

(P2)  $\pi a_{k'} - \pi a_0 \in \Gamma$ ,  $a_{k'} \in S$ .

We note that a subset of a connected set is *not* necessarily connected.

Let

$$\mathcal{S} = \text{supp } u^{(0)} \oplus \text{supp } \bar{u}^{(0)} \subset \mathcal{C}_+ \oplus \mathcal{C}_- = \mathcal{C}. \quad (2.11)$$

We consider the connected sets on (contained in) the bi-characteristics  $\mathcal{C}$ . Below is the main result of the section:

**Proposition 2.1.** *Assume  $u^{(0)}$  is good satisfying the non-degeneracy conditions (i-iii). On the bi-characteristics hyperboloids  $\mathcal{C}$ , the connected sets are of sizes at most  $\max(2d, 2b)$ . If  $b \geq d + 1$ , then the set  $\mathcal{S}$  is the only connected set of size  $2b$ , all the other connected sets are of sizes at most  $2d$ .*

*Proof.* If  $(n, j) \in \mathcal{C}$ , then

$$(n \cdot \omega^{(0)})^2 - j^2 - 1 = 0. \quad (2.12)$$

Therefore, as noted in (1.11),  $n$  must be of the form  $n = n_k e_k$  for some  $k \in \{1, 2, \dots, b\}$ . So the bi-characteristics  $\mathcal{C}$  only consists of “singletons”, i.e.,

$$\mathcal{C} \subseteq \{(n, j) | n = n_k e_k \text{ for some } k = 1, 2, \dots, b, n_k \in \mathbb{Z}\}. \quad (\ddagger)$$

Combined with the definition of a connected set and its ensuing properties (P1, 2), it then follows that (after designating a root) only  $(\nu, \eta)$  of the form considered in condition (iii) could possibly lead to connected sets on  $\mathcal{C}$ .

Assume that there is a connected set  $S$  on the hyperboloid  $\mathcal{C}$ . There are two cases: (a) For all  $(n, j) \in S$ ,  $|n| > p$ , (b) there exists  $(n, j) \in S$  with  $|n| \leq p$ .

Case (a): The size of  $S$  must satisfy

$$|S| \leq 2. \quad (2.13)$$

This is because if  $|S| \geq 3$ , then there must be a connected subset  $S' \subseteq S \subset \mathcal{C}$  satisfying  $|S'| = 3$ . Let  $(n, j)$ ,  $(n', j')$  and  $(n'', j'')$  be the 3 distinct points in  $S'$ , then  $n$ ,  $n'$  and  $n''$  must be of the form  $n = n_k e_k$ ,  $n' = n'_k e_k$  and  $n'' = n''_k e_k$  for some  $k = 1, 2, \dots, b$ . Here we used that for  $(\nu, \eta) \in \Gamma$ ,  $|\nu| \leq p$  and property  $(\dagger\dagger)$ . Call  $(n, j)$  the root.

Let

$$(\nu_1, \eta_1) = (n' - n, j' - j) = ((n'_k - n_k)e_k, -(n'_k - n_k)j_k)$$

and

$$(\nu_2, \eta_2) = (n'' - n, j'' - j) = ((n''_k - n_k)e_k, -(n''_k - n_k)j_k).$$

Subtracting (2.12) evaluated at  $(n', j')$  from the equation at  $(n, j)$  and likewise (2.12) evaluated at  $(n'', j'')$  from that at  $(n, j)$  lead to a system of two linear equations in  $(n, j)$ :

$$\begin{cases} -2(n \cdot \omega^{(0)})(\nu_1 \cdot \omega^{(0)}) + 2j \cdot \eta_1 + \eta_1^2 - (\nu_1 \cdot \omega^{(0)})^2 = 0, \\ -2(n \cdot \omega^{(0)})(\nu_2 \cdot \omega^{(0)}) + 2j \cdot \eta_2 + \eta_2^2 - (\nu_2 \cdot \omega^{(0)})^2 = 0. \end{cases} \quad (2.14)$$

After a straight forward computation, they take the form (\*) with  $m = n_k$ ,  $\ell = k$ ,  $(m_\ell)_1 = (m_k)_1 = -n'_k$  and  $(m_\ell)_2 = (m_k)_2 = -n''_k$ . These two equations are incompatible if  $n'_k \neq n''_k$ , as observed previously in (2.7). So  $|S| < 3$ .

Case (b): Assume  $|S| > 2d$ , then it must contain a *connected* subset  $S'$ ,

$$S' \subseteq S, |S'| = 2d + 1.$$

Subtracting (2.12) evaluated at  $(n', j')$  from the equation evaluated at  $(n, j)$  for all  $(n', j') \in S' \setminus \{(n, j)\}$ , since  $\eta = (n' - n, j' - j) \neq 0$  from condition (iii), there are  $|S'| - 1$  *proper*  $d$ -dimensional hyper-planes of the form in (\*). If the non-degeneracy condition (iii) is applicable, then  $|S'| \leq 2d$ , which is a contradiction.

If (iii) is violated, therefore necessarily  $b \geq d+1$  by using (2.7, 2.8), the only possible solutions  $(n, j)$  form the set

$$M = \{(-me_k, mj_k); |m| = 1, 2, \dots, p\}_{k=1}^b,$$

using the second part of condition (i) and ( $\ddagger$ ). Since

$$M \cap \mathcal{C} = \mathcal{S},$$

the subset restricted to  $|m| = 1$ , the only connected set of size  $2b$  is the exceptional set  $\mathcal{S}$  defined in (2.11).  $\square$

*Remark.* It is important to note that the system of linear equations in (2.14) are in the variables  $n$  and  $j$ . After eliminating the variables  $n$ , it generally leads to *quadratic* polynomials in  $j$ , cf. sect. 5.1. (Here due to the very special property ( $\ddagger$ ), exceptionally, the system is linear in  $j$ .) This is an essential complication compared to NLS, which mostly leads to linear systems in  $j$ , cf. sect. 2.3 of [W1].

### 3. Extraction of parameters

The nonlinear matrix equations in (2.2) are solved using the Lyapunov-Schmidt decomposition, as mentioned in sect. 1. Writing (2.2) collectively as

$$F(u, \bar{u}) = 0,$$

the  $Q$ -equations are the restrictions to the set  $\mathcal{S}$  defined in (2.11):

$$F(u, \bar{u})|_{\mathcal{S}} = 0;$$

the remaining equations are the  $P$ -equations. These equations are solved iteratively using a Newton scheme similar to the one in [W1].

It is natural to start with the  $Q$ -equations and solve for the frequencies:

$$\omega_k = \sqrt{j_k^2 + 1} + \frac{1}{a_k \sqrt{j_k^2 + 1}} \left( \frac{u + \bar{u}}{2} \right)^{*p+1} (-e_k, j_k) + \frac{1}{a_k \sqrt{j_k^2 + 1}} \hat{H}(-e_k, j_k),$$

where  $k = 1, 2, \dots, b$  and  $\hat{H}$  is the Fourier transform of the  $H$  in (1.1). For the first iteration, setting  $u = u^{(0)}$ , we obtain

$$\omega_k^{(1)} = \sqrt{j_k^2 + 1} + \frac{1}{2^{p+1} a_k \sqrt{j_k^2 + 1}} (u^{(0)} + \bar{u}^{(0)})^{*p+1} (-e_k, j_k) + \mathcal{O}(\delta^{p+1}),$$

where  $k = 1, 2, \dots, b$ . So the frequency modulation:

$$\begin{aligned} \Delta\omega_k^{(1)} &:= \omega_k^{(1)} - \omega_k^{(0)} = \frac{1}{2^{p+1} a_k \sqrt{j_k^2 + 1}} (u^{(0)} + \bar{u}^{(0)})^{*p+1} (-e_k, j_k) + \mathcal{O}(\delta^{p+1}) \\ &:= \Omega_k + \mathcal{O}(\delta^{p+1}), \end{aligned} \tag{3.1}$$

where  $k = 1, 2, \dots, b$ .

There are the following estimates on amplitude-frequency modulation.

**Proposition 3.1.** *Assume that  $u^{(0)} = \sum_{k=1}^b a_k e^{-i(\sqrt{j_k^2+1})t} e^{ij_k \cdot x}$  is a solution to the linear equation with  $b$  frequencies and  $a = \{a_k\}_{k=1}^b \in (0, \delta]^b = \mathcal{B}(0, \delta) = \mathcal{B} \subset \mathbb{R}^b \setminus \{0\}$ . Assume  $p$  is even and  $\epsilon' \in (0, 1)$ . There exists  $\delta_0 > 0$  such that for all  $\delta \in (0, \delta_0)$  and  $a \in \mathcal{B}$ ,*

$$\begin{aligned} \|\Delta\omega^{(1)}\| &\asymp \delta^p, \\ \left\| \frac{\partial\omega^{(1)}}{\partial a} \right\| &\asymp \delta^{p-1}, \end{aligned}$$

where the constants implied by  $\asymp$  depend only on  $p, b$  and  $d$ ; further there exists a subset  $\mathcal{B}' \subset \mathcal{B}$  with

$$\text{meas } \mathcal{B}' < \epsilon' \delta^b / 2,$$

such that if  $a \in \mathcal{B} \setminus \mathcal{B}'$ , an open set, then

$$\begin{aligned} \left\| \left( \frac{\partial\omega^{(1)}}{\partial a} \right)^{-1} \right\| &\lesssim \delta^{-p+1}, \\ \left| \det \left( \frac{\partial\omega^{(1)}}{\partial a} \right) \right| &\gtrsim \delta^{(p-1)b}, \end{aligned}$$

where the constants implied by  $\lesssim$  and  $\gtrsim$  only depend on  $p, b, d$  and  $\epsilon'$ .

*Proof.* Let  $M$  be the convolution matrix

$$M = (u^{(0)} * \bar{u}^{(0)})^{*p/2} *, \quad (3.2)$$

and

$$D_k = 2^{p+1} \sqrt{j_k^2 + 1}, \quad k = 1, 2, \dots, b.$$

Then using (3.1) and expanding the  $(p+1)$ -fold convolution, we obtain that

$$\begin{aligned} B_k &:= D_k \Omega_k \\ &= C_{p+1}^{p/2} (M_{kk} + \sum_{i \neq k} \frac{M_{ki} a_i}{a_k}), \quad k = 1, \dots, b, \end{aligned} \quad (3.3)$$

cf. the proof of Proposition 3.3 in sect. 3.2 of [W1]. Here we used that

$$(u^{(0)} * \bar{u}^{(0)})^{*p/2} * u^{(0)}$$

is the only contributing term. (This is the same term that appears in the NLS in [W1] and we shall use some of its properties derived there.)

From the structure of  $M$ ,

$$M_{kk}(a_1, a_2, \dots, a_b) = P(a_1, a_2, \dots, a_b)$$

and

$$M_{ki} = P'(a_1, a_2, \dots, a_b) a_k \bar{a}_i, \quad k \neq i$$

where  $P$  and  $P'$  are homogeneous polynomials in  $a$  with positive integer coefficients and are invariant under any permutations of the arguments,  $P$  is of degree  $p$ ,  $P'$ ,  $(p-2)$ . So  $B_k$  is a homogeneous polynomial in  $\{a_i\}_{i=1}^b$  of degree  $p$  and can be written as

$$B_k(a_1, a_2, \dots, a_b) = C_{p+1}^{p/2} [P(a_1, a_2, \dots, a_b) + P'(a_1, a_2, \dots, a_b) P_k(\{a_i\}_{i \neq k})],$$

with  $P_k = \sum_{i \neq k} a_i^2$  and  $P, P'$  as above, for  $k = 1, 2, \dots, b$ . Since  $\Omega_k = B_k/D_k$ ,  $k = 1, 2, \dots, b$ , this yields the estimate on  $\|\Delta \omega^{(1)}\|$  using that  $P, P'$  and  $P_k$  are polynomials with positive coefficients and that  $a_k > 0$  for all  $k = 1, 2, \dots, b$ . Similar arguments produces the second estimate.

Setting  $a = (1, 1, \dots, 1)$ , we therefore have

$$\frac{\partial B_k}{\partial a_i}(1, 1, \dots, 1) > \frac{\partial B_k}{\partial a_k}(1, 1, \dots, 1)$$

for all  $i \neq k$ . Let  $q$  be the diagonal elements and  $Q$  the off-diagonal ones at  $(1, 1, \dots, 1)$ . This gives  $q, Q \in \mathbb{N}^+$  satisfying

$$0 < q < Q.$$

For example, in the cubic case,  $p = 2$ ,  $P = \sum_{i=1}^b a_i^2$ ,  $P' = 1$  and  $P_k = \sum_{i \neq k} a_i^2$  giving  $q = 6$  and  $Q = 12$ .

Using the same argument as in the proof of Proposition 3.3, sect. 3.2 of [W1], the partial derivative matrix:

$$[[\frac{\partial B_k}{\partial a_i}]]$$

has a simple eigenvalue  $\lambda_1 = q + (b-1)Q \neq 0$  and a  $(b-1)$ - fold degenerate eigenvalue  $\lambda_2 = q - Q \neq 0$ . So

$$\det \left( \frac{\partial B_k}{\partial a_i} \right) (1, 1, \dots, 1) \neq 0 = \det \left( \frac{\partial B_k}{\partial a_i} \right) (0, 0, \dots, 0).$$

Hence  $\det \left( \frac{\partial B_k}{\partial a_i} \right)$  is not a constant.

Let  $D$  be the  $b \times b$  diagonal matrix with diagonals  $D_k$ ,  $k = 1, 2, \dots, b$ . Since  $\det \left( \frac{\partial B_k}{\partial a_i} \right)$  is a homogeneous polynomial in  $a$  of degree at most  $(p-1)b$  and  $D$  is bounded and invertible, this proves the last two estimates taking into account also the  $\mathcal{O}(\delta^p)$  perturbation.  $\square$

*Remark.* When  $p$  is odd,  $\Omega_k(a) = 0$ ,  $k = 1, 2, \dots, b$ , for all  $a$ . There is no frequency modulation at order  $\mathcal{O}(\delta^p)$ . As mentioned earlier, this is why  $p$  is taken to be even. Note also that Proposition 3.1 does not need  $u^{(0)}$  to be good, as it is *independent* of the correction  $\Delta u^{(1)}$ .

#### 4. The first step

Let  $F'$  be the operator linearized at  $u^{(0)}, \bar{u}^{(0)}$  and evaluated at  $\omega = \omega^{(1)}$ . Let  $F'_N(\omega^{(1)}, u^{(0)}, \bar{u}^{(0)})$  be the restricted operator:

$$\begin{aligned} F'_N(n, j; n', j') &= F'(n, j; n', j'), & \text{if } (n, j), (n', j') \in [-N, N]^{b+d} \times \{0, 1\} \setminus \mathcal{S}, \\ &= 0, & \text{otherwise,} \end{aligned}$$

where  $\mathcal{S}$  is as defined in (2.11). In other words, the operator  $F'_N$  is  $F'$  restricted to the intersection of the domain of the  $P$ -equations with  $[-N, N]^{b+d} \times \{0, 1\}$ . (Recall that for simplicity, the  $\{0, 1\}$  index is generally omitted when writing a matrix element.) We make the first corrections to the unperturbed solution  $u^{(0)}$  by solving the  $P$ -equations, using a Newton scheme. The key is the invertibility of  $F'_N$ . For the first iteration, we take

$$N = |\log \delta|^s, \tag{4.1}$$

for some  $s > 1$  to be determined later in (5.33) of Lemma 5.7.



**Lemma 4.1.** Assume that  $u^{(0)}$  is good satisfying the non-degeneracy conditions (i-iii) and  $p$  even. Assume that  $b > C_p d$ , where  $C_p$  only depends on  $p$ . Let  $\epsilon \in (0, 1)$ . There exists  $\delta_0 > 0$  such that for all  $\delta \in (0, \delta_0)$ , there exists a subset  $\mathcal{B}_{\epsilon', \epsilon, \delta}$ ,  $(0, \delta]^b := \mathcal{B} \supset \mathcal{B}_{\epsilon', \epsilon, \delta} \supset \mathcal{B}'$ , the set in Proposition 3.1, with

$$\text{meas } \mathcal{B}_{\epsilon', \epsilon, \delta} < \epsilon' \delta^b.$$

On  $\mathcal{B} \setminus \mathcal{B}_{\epsilon', \epsilon, \delta}$ , an open subset, the operator  $F'_N$  satisfies:

$$\|[F'_N]^{-1}\| \leq \delta^{-p-\epsilon}, \quad (4.2)$$

and there exists  $\beta \in (0, 1)$ , depending only on  $\text{supp } u^{(0)}$ ,  $p, b, d$  and the  $H$  in (1.1) such that

$$|[F'_N]^{-1}(x, y)| \leq \delta^{\beta|x-y|} = e^{-\beta|\log \delta||x-y|} \quad (4.3)$$

for all  $x, y \in [-N, N]^{b+d} \times \{0, 1\} \setminus \mathcal{S}$  such that  $|x-y| > 1/\beta^2$ , where  $|x-y| := |\pi x - \pi y|$  and  $\pi$  is the projection onto  $\mathbb{Z}^{b+d}$ .

Let  $P_{\pm}$  be the projection on  $\mathbb{Z}^{b+d}$  onto  $\mathcal{C}_{\pm}$  defined in (1.8),

$$P = \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix}. \quad (4.4)$$

The following plays a key role toward proving Lemma 4.1.

**Lemma 4.2.** Assume that  $u^{(0)}$  is good. The linearized operator  $F'$  evaluated at  $\omega^{(1)}$ ,  $u^{(0)}$ ,  $\bar{u}^{(0)}$  and restricted to  $\mathcal{C} \setminus \mathcal{S} \subset \mathbb{Z}^{b+d} \times \{0, 1\}$ :  $PF'P$  can be written as

$$PF'P = \oplus_{\alpha} \mathcal{F}_{\alpha} + \mathcal{O}(\delta^{p+1}),$$

where  $\|\mathcal{F}_{\alpha}\| = \mathcal{O}(\delta^p)$ ,  $\alpha$  are connected sets on  $\mathcal{C} \setminus \mathcal{S}$  satisfying  $|\alpha| \leq 2d$ ;

$$\mathcal{F}_{\alpha} = \begin{pmatrix} \text{diag } (n \cdot \Omega \sqrt{j^2 + 1}) & 0 \\ 0 & \text{diag } (-n \cdot \Omega \sqrt{j^2 + 1}) \end{pmatrix} \Big|_{\alpha} + A_{\alpha}, \quad (4.5)$$

$\Omega = \{\Omega_k\}_{k=1}^b$  as defined in (3.1),  $(n, j) \in \alpha$ ,  $A_{\alpha}$  is the  $A$  defined in (2.5) restricted to  $\alpha$ .  $\mathcal{F}_{\alpha}$  are therefore matrices of sizes at most  $2d \times 2d$ . Furthermore  $\mathcal{F}_{\alpha}$  satisfies

$$\det \mathcal{F}_{\alpha} \neq 0,$$

if  $b > C_{p, \alpha} d$ .

*Proof.* Each block of  $PAP$  generates a maximally connected set (a connected set which is maximal in size)  $\alpha$  on  $\mathcal{C} \setminus \mathcal{S}$ . The size of the block, is the size of the maximally

connected set  $\alpha$ . So  $|\alpha| \leq 2d$ , using Proposition 2.1. Equations (2.4, 3.1) are then used to arrive at the first matrix in (4.5).

To prove the second part of the Lemma, we set

$$a_1 = a_2 = \dots = a_b = 1.$$

It is essentially a direct computation similar to the proof of Lemma 4.2 in [W1] and we shall use some of the derivations there.

We first note that if  $(n, j) \in \mathcal{C}$ , then  $n = n_k e_k$  for some  $k = 1, 2, \dots, b$ , and  $n_k, j$  satisfy

$$(n_k \omega_k^{(0)})^2 - j^2 - 1 = 0.$$

From (2.4) with  $\omega = \omega^{(1)}$ , (2.5, 3.1, 3.3), it then follows that

$$2^{p+1} \mathcal{F}_\alpha(n, j; n, j) = -n_k^2 C_{p+1}^{p/2} (M_{11} + (b-1)M_{12}) + (p+1)C_p^{p/2} M_{11},$$

where  $M_{11}, M_{12}$  denote respectively the  $(1, 1), (1, 2)$  element of the matrix  $M$  defined in (3.2) and we used the symmetry:  $M_{ii} = M_{11}$  for all  $i$  and  $M_{ij} = M_{12}$  for all  $i, j, i \neq j$ .  $M_{11}$  is a polynomial in  $b$  of degree  $p/2$ ; while  $M_{12}, p/2 - 1$ .

From [W1], with  $p \rightarrow p/2$ , the combinatorial factor, the coefficient, in front of the  $\mathcal{O}(b^{p/2})$  term in  $M_{11}$  is

$$1 + 2!C_{p/2}^2 + 3!C_{p/2}^3 + \dots + m!C_{p/2}^m + \dots + (p/2)!C_{p/2}^{p/2};$$

while the coefficient in front of the leading order  $\mathcal{O}(b^{p/2-1})$  term for  $M_{12}$  is

$$C_{p/2}^1 + 2!C_{p/2}^2 + 3!C_{p/2}^3 + \dots + m!C_{p/2}^m + \dots + (p/2)!C_{p/2}^{p/2}.$$

These two formulae appear just above (4.14) in [W1] and are derived using the binomial expansion.

Write

$$2^{p+1} \mathcal{F}_\alpha(n, j; n, j) = Rb^{p/2} + \mathcal{O}(b^{p/2-1}).$$

Using the above two formulae, it follows from direct computation that when  $p = 2$ ,  $R = 0$  if and only if  $n_k = \pm 1$ . When  $p > 2$ , write  $N = n_k^2$ . Setting  $R = 0$  leads to

$$\begin{aligned} N &= \frac{(p/2 + 1)(\sum_{m=2}^{p/2} m!C_{p/2}^m + 1)}{(p/2 + 1) + 2 \sum_{m=2}^{p/2} m!C_{p/2}^m} \\ &= \left(\frac{p+2}{4}\right) \left(\frac{1 + \frac{1}{A}}{1 + \frac{p+2}{4A}}\right), \\ &= \left(\frac{p+2}{4}\right) + \left(\frac{p+2}{4}\right) \left[\left(1 + \frac{1}{A}\right) \sum_{n=1}^{\infty} (-x)^n + \frac{1}{A}\right] \end{aligned}$$

where  $p > 2$ ,  $A = \sum_{m=2}^{p/2} m! C_{p/2}^m$  and  $x = \frac{p+2}{4A}$ .

Since

$$0 < \left| \left( \frac{p+2}{4} \right) \left[ \left( 1 + \frac{1}{A} \right) \sum_{n=1}^{\infty} (-x)^n + \frac{1}{A} \right] \right| < 1/2$$

from direct computation,  $N \notin \mathbb{Z}$  for  $p > 2$ . So  $R \neq 0$  for integer  $n_k^2$ . Taking into account that  $(\mp e_k, \pm j_k)$ ,  $k = 1, 2, \dots, b$ , are in  $\mathcal{S}$ , this proves that

$$\mathcal{F}_\alpha(n, j; n, j) = \mathcal{O}(b^{p/2}),$$

for all  $(n, j) \in \alpha$  on  $\mathcal{C} \setminus \mathcal{S}$ . Since all off-diagonal elements are of order at most  $\mathcal{O}(b^{p/2-1})$  and  $\mathcal{F}_\alpha$  is at most of size  $2d \times 2d$ ,

$$\det \mathcal{F}_\alpha(1, 1, \dots, 1) \neq 0$$

for  $b > C_{p,\alpha}d$ ; so

$$\det \mathcal{F}_\alpha(1, 1, \dots, 1) \neq 0.$$

□

#### 4.1. Proof of Lemma 4.1.

The proof is rather lengthy. It is therefore separated into two parts. We first prove the norm estimate.

*Proof of (4.2) of Lemma 4.1.* One first notes that if

$$\pm n \cdot \omega^{(0)} + \sqrt{j^2 + 1} \neq 0,$$

then

$$|\pm n \cdot \omega^{(0)} + \sqrt{j^2 + 1}| \geq c' \|n\|_1^{-q}, \quad (4.6)$$

for some  $c' > 0$  and  $q > b^2$ , using [Schm], cf. also [R] for the scalar case. This follows from multiplying the two expressions corresponding to the  $\pm$  signs in the absolute value and  $|j| \lesssim |n|$  (otherwise the inequality in (4.6) is trivially true). The bound in (4.6) implies that

$$\|[P^c F'_N P^c]^{-1}\| \leq C' N^q, \quad (4.7)$$

where  $P^c = I - P$ , for some  $C' > 0$  and small  $\delta$ .

From Schur's complement reduction [S1, 2],  $\lambda$  is in the spectrum of  $F'_N$ ,  $\lambda \in \sigma(F'_N)$ , if and only if  $0 \in \sigma(\mathcal{H})$ , where

$$\mathcal{H} = P F'_N P - \lambda + P F'_N P^c (P^c F'_N P^c - \lambda)^{-1} P^c F'_N P. \quad (4.8)$$

Moreover (4.7) implies that (4.8) is analytic in  $\lambda$  in the interval

$$(-1/(2C'N^q), 1/(2C'N^q))$$

and on the same interval

$$\|PF'_N P^c (P^c F'_N P^c - \lambda)^{-1} P^c F'_N P\| \leq \mathcal{O}(\delta^{2p} N^q).$$

So for small  $\delta$ , we only needs to prove invertibility of the first term in (4.8), which is a much smaller matrix.

Toward that purpose, we identify the set of connected sets  $\{\alpha\}$  on

$$\mathcal{C} \cap [-N, N]^{b+d} \times \{0, 1\} \setminus \mathcal{S}$$

with the set  $\{1, 2, \dots, K_1\}$ , where  $K_1 = K_1(N)$ . So

$$PF'_N P = \oplus_k \mathcal{F}_k(a) + \mathcal{O}(\delta^{p+1}), \quad k \leq K_1(N),$$

where each  $\mathcal{F}_k$  is of the form in (4.5).

Fix

$$N_0 = N_0(p, b, d, \epsilon') \tag{4.9}$$

large to be determined by (4.10) below. For a given  $\mathcal{F}_k$ , define the support of  $\mathcal{F}_k$  to be

$$\mathbb{Z}^{b+d} \times \mathbb{Z}^{b+d} \supset \text{supp } \mathcal{F}_k = \{(\pi x, \pi y) | \mathcal{F}_k(x, y) \neq 0\},$$

where  $\pi$  is the projection onto  $\mathbb{Z}^{b+d}$ .

For matrices  $\mathcal{F}_k$ , such that

$$\text{supp } \mathcal{F}_k \cap \{[-N_0, N_0]^{b+d} \times [-N_0, N_0]^{b+d}\} \neq \emptyset,$$

we use the determinant. There are at most  $K_0$  (independent of  $\delta$ ) of these matrices. Let

$$w = a\delta^{-1},$$

and

$$P_k = P_k(a) := \det \mathcal{F}_k(a\delta^{-1}) = \det \mathcal{F}_k(w) \neq 0,$$

using Lemma 4.1. Therefore there exist  $C, c > 0$ , such that given  $\epsilon \in (0, 1)$ , there exists  $\delta_0 \in (0, 1)$ , such that for all  $\delta \in (0, \delta_0)$ ,

$$\text{meas } \{a \in \mathcal{B} | |P_k| < \delta^\epsilon, \text{ all } k \leq K_0\} \leq C\delta^{b+c\epsilon}.$$

So  $\|\mathcal{F}_k^{-1}(a)\| \leq \mathcal{O}(\delta^{-p-\epsilon})$  for all  $k \leq K_0$ .

For matrices  $\mathcal{F}_k$  with  $k > K_0$ ,

$$\text{supp } \mathcal{F}_k \cap \{[-N_0, N_0]^{b+d} \times [-N_0, N_0]^{b+d}\} = \emptyset$$

by definition. We use perturbation theory. This corresponds to case a) in the proof of Proposition 2.1. So  $\mathcal{F}_k$  is at most a  $2 \times 2$  matrix and from (§) there is  $\ell \in \{1, 2, \dots, b\}$  such that  $n \cdot \Omega = n_\ell \Omega_\ell$ . Moreover

$$|\frac{\partial^2}{\partial \Omega_\ell^2} P_k| + |\frac{\partial}{\partial \Omega_\ell} P_k| > \frac{1}{2} \quad (4.10)$$

for  $N_0$  large enough depending only on  $p, b, d$  and  $\epsilon'$ , where we used the form of the matrix in (4.5) and

$$\|\frac{\partial \Omega}{\partial w}\| \asymp \|(\frac{\partial \Omega}{\partial w})^{-1}\| \asymp \mathcal{O}(1),$$

from Proposition 3.1 with the constants implied by  $\asymp$  depending only on  $p, b, d$  and  $\epsilon'$ , and that there are only finite types of “convolution” matrices  $A_k$ .

Since  $\|\mathcal{F}_k(w)\| \leq \mathcal{O}(|\log \delta|^{2s})$ , (4.10) gives  $\|\mathcal{F}_k(w)^{-1}\| \leq \mathcal{O}(\delta^{-\epsilon})$  for all  $K_0 < k \leq K_1(N)$  away from a set in  $w$  of measure less than  $\epsilon'/2 + \delta^{\epsilon/3}$ , where we also used  $K_1(N) \leq \mathcal{O}(|\log \delta|^{2(b+d)s})$ ,  $s > 1$ . So  $\|\mathcal{F}_k(a)^{-1}\| \leq \mathcal{O}(\delta^{-p-\epsilon})$  away from a set in  $a$  of measure less than  $(\epsilon'/2 + \delta^{\epsilon/3})\delta^b$ .

Combining the above two regions, one has that away from a set in  $a$  of measure less than  $\epsilon'\delta^b$ ,

$$\|[PF'_N(a)P]^{-1}\| \leq \mathcal{O}(\delta^{-p-\epsilon}).$$

The Schur reduction in (4.8) then gives (4.2).  $\square$

*Proof of (4.3) of Lemma 4.1.* To obtain the point-wise estimates, let  $w = a\delta^{-1}$  as before and

$$\begin{aligned} \tilde{F} &= \oplus_\alpha \delta^p \mathcal{F}_\alpha \oplus \text{diag} [(n \cdot \omega^{(1)} + \sqrt{j^2 + 1})\sqrt{j^2 + 1} + \delta^p A(n, j; n, j)]|_{(n, j) \notin \mathcal{C}^+} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad \oplus \text{diag} [(-n' \cdot \omega^{(1)} + \sqrt{j'^2 + 1})\sqrt{j'^2 + 1} + \delta^p A(n', j'; n', j')]|_{(n', j') \notin \mathcal{C}^-} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\ &:= \oplus_\alpha \delta^p \mathcal{F}_\alpha \oplus \mathcal{D}_{\mathbb{Z}^{b+d} \times \mathbb{Z}_2 \setminus \mathcal{C}} \end{aligned} \quad (4.11)$$

where the first direct sum is exactly as in Lemma 4.2, with  $\alpha$  connected subsets of  $\mathcal{C}$ ,  $\tilde{F} := \tilde{F}(w)$ ,  $\mathcal{F}_\alpha := \mathcal{F}_\alpha(w)$  and  $A := A(w)$  as in (2.5).

For simplicity, write  $x$  for  $(n, j)$  etc. The matrix  $A$  is a convolution matrix with diagonal  $\bar{A}$ . Let  $\Lambda_1$  be the matrix such that

$$\begin{aligned} \Lambda_1(x, y) &= \Lambda_1(y, x) = (A - \bar{A}I)(x, y), \text{ if } x \in \mathcal{C}, y \in \mathbb{Z}^{b+d} \times \{0, 1\} \setminus \mathcal{C}, \\ &= 0 \quad \text{otherwise,} \end{aligned} \quad (4.12)$$

$\|\Lambda_1\| = \mathcal{O}(\delta^p)$ . Write

$$F' = \tilde{F} + \Lambda = \tilde{F} + \Lambda_1 + \Lambda_2,$$

with  $\|\Lambda_2\| = \mathcal{O}(\delta^{p+1})$ . The resolvent expansion gives:

$$[F'_N]^{-1} = [\tilde{F}_N]^{-1} - [\tilde{F}_N]^{-1}\Lambda_N[\tilde{F}_N]^{-1} + [\tilde{F}_N]^{-1}\Lambda_N[\tilde{F}_N]^{-1}\Lambda_N[F'_N]^{-1}, \quad (4.13)$$

where as before the sub-index  $N$  denotes the restriction to  $[-N, N]^{b+d} \times \{0, 1\}$ . The analysis of the series is similar to the proof of (3.3) in Lemma 3.1 in [W1]. Below we summarize the key steps. For simplicity of notation, the subscript  $N$  is omitted.

We estimate the second term in the expansion:

$$[\tilde{F}]^{-1}\Lambda[\tilde{F}]^{-1} = [\tilde{F}]^{-1}\Lambda_1[\tilde{F}]^{-1} + [\tilde{F}]^{-1}\Lambda_2[\tilde{F}]^{-1}. \quad (4.14)$$

Since  $\|\Lambda_2\| = \mathcal{O}(\delta^{p+1})$ ,

$$\|[\tilde{F}]^{-1}\Lambda_2[\tilde{F}]^{-1}\| \leq \mathcal{O}(\delta^{1-\epsilon})\|[\tilde{F}]^{-1}\| \quad (4.15)$$

using (4.2).

To estimate the first term in the sum in (4.14), we need to estimate  $[\mathcal{F}_\alpha]^{-1}\Lambda_1[\mathcal{F}_{\alpha'}]^{-1}$ ,  $[\mathcal{F}_\alpha]^{-1}\Lambda_1[\mathcal{D}]^{-1}$ ,  $[\mathcal{D}]^{-1}\Lambda_1[\mathcal{F}_{\alpha'}]^{-1}$  and  $[\mathcal{D}]^{-1}\Lambda_1[\mathcal{D}]^{-1}$ . From the definition of  $\Lambda_1$  in (4.12), the first term

$$[\mathcal{F}_\alpha]^{-1}\Lambda_1[\mathcal{F}_{\alpha'}]^{-1} = 0.$$

Using (4.2) and since

$$\|\mathcal{D}^{-1}\| \leq \mathcal{O}(|\log \delta|^{sq})$$

from (4.6) and small  $\delta$ , summing over the last three terms yields

$$\|[\tilde{F}]^{-1}\Lambda_1[\tilde{F}]^{-1}\| \leq \mathcal{O}(\delta^{p-\tilde{\epsilon}})\|[\tilde{F}]^{-1}\|, \quad 0 < \tilde{\epsilon} < 1. \quad (4.16)$$

Iterating the resolvent expansion in (4.13)  $r$  times yields the  $(r+1)$  term series

$$[F']^{-1} = [\tilde{F}]^{-1} - [\tilde{F}]^{-1}\Lambda[\tilde{F}]^{-1} + \dots + (-1)^r[\tilde{F}^{-1}\Lambda]^r[F']^{-1}.$$

We note that the blocks in  $\tilde{F}$  (and hence  $[\tilde{F}]^{-1}$ ) are of sizes at most  $2d$  and that  $\Lambda$  satisfies  $\Lambda(x, y) = 0$  if  $|x - y| > C$ , for some  $C$  depending only on  $\text{supp } u^{(0)}$ ,  $p$ ,  $b$ ,  $d$  and  $H$  in (1.1). Matrix multiplication then infers that for some  $\beta > 0$  depending only on  $\text{supp } u^{(0)}$ ,  $p$ ,  $b$ ,  $d$  and  $H$  and any given  $x, y \in \mathbb{Z}^{b+d} \times \{0, 1\}$  satisfying  $|x - y| > 1/\beta^2$ , there exists  $r > 1$  such that the first  $r$  terms in the series are identically 0. Iterating the bounds in (4.15, 4.16) and using (4.2) to estimate the last, the  $(r+1)$ th term produces (4.3).  $\square$

## 4.2. The first approximate solution.

The nonlinear matrix equation (2.2) is now ready to be solved iteratively. The solution  $u$  is held fixed on  $\mathcal{S}$ :

$$u(\mp e_k, \pm j_k) = a_k, \quad k = 1, \dots, b.$$

The set  $\mathcal{S}$  is the domain of the  $Q$ -equations and is used to solve for the frequencies. The first iteration was solved in sect. 3. To solve the  $P$ -equations, restrict the domain to

$$[-N, N]^{b+d} \times \{0, 1\} \setminus \mathcal{S}, \quad \text{where } N = |\log \delta|^s, \quad s > 1,$$

for the first iteration and define

$$\Delta u^{(1)} := u^{(1)} - u^{(0)} = [F'_N]^{-1}(\omega^{(1)}, u^{(0)}, \bar{u}^{(0)})F(\omega^{(1)}, u^{(0)}, \bar{u}^{(0)}).$$

Below is a summary of the findings.

**Proposition 4.3.** *Assume that  $u^{(0)} = \sum_{k=1}^b a_k e^{-i(\sqrt{j_k^2+1})t} e^{ij_k \cdot x}$  a solution to the linear equation with  $b$  frequencies is good and  $a = \{a_k\} \in (0, \delta]^b = \mathcal{B} \subset \mathbb{R}^b \setminus \{0\}$ . Assume that  $b > C_p d$ , where  $C_p$  only depends on  $p$ . Let  $\epsilon', \epsilon \in (0, 1)$ . There exists  $\delta_0 > 0$ , such that for all  $\delta \in (0, \delta_0)$ , there is a subset  $\mathcal{B}_{\epsilon', \epsilon, \delta} \subset \mathcal{B}$  with*

$$\text{meas } \mathcal{B}_{\epsilon', \epsilon, \delta} < \epsilon' \delta^b.$$

Let  $\rho$  be a weight on  $\mathbb{Z}^{b+d}$  satisfying

$$\begin{aligned} \rho(x) &= e^{\beta |\log \delta| |x|}, \quad 0 < \beta < 1 \quad \text{for } |x| > 1/\beta^2, \\ &= 1, \quad \text{for } |x| \leq 1/\beta^2. \end{aligned}$$

Define the weighted  $\ell^2$  norm:

$$\|\cdot\|_{\ell^2(\rho)} = \|\rho \cdot\|_{\ell^2}.$$

There exists  $\beta \in (0, 1)$ , determined only by  $\text{supp } u^{(0)}$ ,  $p$ ,  $b$ ,  $d$  and  $H$  in (1.1), such that if  $a \in \mathcal{B} \setminus \mathcal{B}_{\epsilon', \epsilon, \delta}$ , an open subset, then

$$\|\Delta u^{(1)}\|_{\ell^2(\rho)} = \|\Delta \bar{u}^{(1)}\|_{\ell^2(\rho)} \lesssim \delta^{2-\epsilon}, \quad (4.17)$$

$$\|F(u^{(1)}, \bar{u}^{(1)})\|_{\ell^2(\rho)} \lesssim \delta^{p+3-2\epsilon}, \quad (4.18)$$

$$\|\Delta \omega^{(1)}\| \asymp \delta^p, \quad (4.19)$$

$$\left\| \frac{\partial \omega^{(1)}}{\partial a} \right\| \asymp \delta^{p-1}, \quad (4.20)$$

$$\left\| \left( \frac{\partial \omega^{(1)}}{\partial a} \right)^{-1} \right\| \lesssim \delta^{-p+1}, \quad (4.21)$$

$$\left| \det \left( \frac{\partial \omega^{(1)}}{\partial a} \right) \right| \gtrsim \delta^{(p-1)b}, \quad (4.22)$$

where  $\omega^{(1)}$  and  $\Delta\omega^{(1)}$  as defined in (3.1); the implied constants in (4.17, 4.21, 4.22) depend on  $p, b, d$  and  $\epsilon'$ ; while that in (4.18-4.20), only on  $p, b$  and  $d$ . Moreover  $\omega^{(1)}$  is Diophantine

$$\|n \cdot \omega^{(1)}\|_{\mathbb{T}} \geq \frac{\xi}{|n|^\gamma}, \quad n \in [-N, N]^b \setminus \{0\}, \quad \xi > 0, \gamma > 2b, \quad (4.23)$$

where  $\|\cdot\|_{\mathbb{T}}$  denotes the distance to integers in  $\mathbb{R}$  and  $\xi$  only depends on  $\omega^{(0)}$ .

*Proof.* We only need to prove (4.17, 4.18, 4.23). The rest is the content of (proven) Proposition 3.1. Write  $F(u^{(0)})$  as

$$F(u^{(0)}) := F_0(u^{(0)}) + \mathcal{O}(\delta^{p+2}),$$

where

$$F_0(u^{(0)}) = \left( \frac{u^{(0)} + \bar{u}^{(0)}}{2} \right)^{*(p+1)}.$$

The support of  $F_0$ ,  $\text{supp } F_0 = \{(\nu, \eta)\}$ , consists of  $(\nu, \eta)$  of the form

$$(\nu, \eta) = \left( -\sum_{k=1}^b m_k e_k, \sum_{k=1}^b m_k j_k \right),$$

with  $\sum_{k=1}^b |m_k| \leq p+1$ . Using the non-degeneracy condition (ii) (property  $(\dagger\dagger)$ ) and that

$$(-\ell e_k, \ell j_k) \notin \mathcal{C}, \quad k = 1, 2, \dots, b,$$

for  $|\ell| \neq 1$ , it follows that

$$F_0(u^{(0)}) \cap \mathcal{C} \setminus \mathcal{S} = \emptyset.$$

From the Newton scheme

$$\begin{aligned} \Delta \begin{pmatrix} u^{(1)} \\ \bar{u}^{(1)} \end{pmatrix} &= -[F'_N(u^{(0)}, \bar{u}^{(0)})]^{-1} F(u^{(0)}, \bar{u}^{(0)}) \\ &= -(F'_N)^{-1} F_0 + \mathcal{O}(\delta^{2-\epsilon}) \\ &= -\mathcal{D}^{-1} F_0 + (F'_N)^{-1} (F'_N - \mathcal{D}) \mathcal{D}^{-1} F_0 + \mathcal{O}(\delta^{2-\epsilon}) \\ &\sim |\log \delta|^{sq} \delta^{p+1} (1 + \delta^{-\epsilon}) + \mathcal{O}(\delta^{2-\epsilon}) \\ &= \mathcal{O}(\delta^{2-\epsilon}) \end{aligned}$$

in  $\ell^2$  norm, where we used (4.2),  $\mathcal{D}$  as defined in (4.11), we used (4.6) and small  $\delta$ . Using the point-wise estimate (4.3), the (at least) exponential off-diagonal decay of



$F'_N$ , and since  $\text{supp } F$  is a compact set, the above bound remains valid in the weighted  $\ell^2$  norm,  $\|\cdot\|_{\ell^2(\rho)}$  and we obtain (4.17).

Since the  $Q$ -equations are solved exactly,  $F_{\mathcal{S}}(u + \Delta u) = 0$ . (For simplicity, the superscripts have been dropped.) To prove (4.18), we only need to be concerned with  $F_{\mathcal{S}^c}(u + \Delta u)$ . Below (and in general) for notational simplicity, we omit the subscript  $\mathcal{S}^c$  and write

$$\begin{aligned} F(u + \Delta u) &= F(u) + F'(u)\Delta u + \mathcal{O}(\|F''(u)\|\|\Delta u\|^2), \\ &= (F' - F'_N)[F'_N]^{-1}F(u) + \mathcal{O}(\|F''(u)\|\|\Delta u\|^2), \end{aligned}$$

where  $u$  stands for  $\begin{pmatrix} u^{(0)} \\ \bar{u}^{(0)} \end{pmatrix}$ ,  $\Delta u$  stands for  $\begin{pmatrix} \Delta u^{(1)} \\ \Delta \bar{u}^{(1)} \end{pmatrix}$ . Since

$$\begin{aligned} [F' - F'_N](x, y) &= 0, & x, y \in [-N, N]^{b+d} \times \{0, 1\} \setminus \mathcal{S}, \\ &= F'(x, y), & \text{otherwise,} \end{aligned}$$

using the exponential off-diagonal decay of  $F'$ , (4.3), compactness of  $\text{supp } F$  and (4.17), we obtain

$$\|F(u + \Delta u)\|_{\ell^2(\rho)} = \mathcal{O}(\delta^{p+3-2\epsilon}).$$

The (linear) Diophantine property in (4.23) is a consequence of  $(\dagger)$  in sect. 2.1, using [Schm], small  $\delta$  and (4.1).  $\square$

## 5. The $\theta$ estimates

Proposition 4.3 puts the construction in a non-resonant form with  $\omega^{(1)}$  as the parameter. It provides the input for the initial scales in the Newton scheme. To continue the iteration, we need the analogues of Lemma 4.1 and Proposition 4.3 at larger scales. This section prepares the way toward achieving that goal.

Let  $T = F'$  be the linearized operator defined as in (2.3-2.5) and the restricted operator  $T_N = F'_N$  as defined above Lemma 4.1. To increase the scale from  $N$  to a larger scale  $N_1$ , we pave the  $N_1$  cubes with  $N$  cubes. We add a one dimensional parameter  $\theta \in \mathbb{R}$  and consider  $T(\theta)$ :

$$\begin{aligned} T(\theta) &= \begin{pmatrix} \text{diag} [\sqrt{j^2 + 1}(n \cdot \omega + \sqrt{j^2 + 1} + \theta)] & 0 \\ 0 & \text{diag} [\sqrt{j^2 + 1}(-n \cdot \omega + \sqrt{j^2 + 1} - \theta)] \end{pmatrix} \\ &\quad + A + \mathcal{O}(\delta^{p+1}) \\ &= D'(\theta) + A + \mathcal{O}(\delta^{p+1}), \end{aligned} \tag{5.1}$$

where  $\omega$  standing for  $\omega^{(1)}$ , is considered as a parameter in this section,  $A$  as in (2.5) and the  $\mathcal{O}(\delta^{p+1})$  as in (2.3).

We note that this one dimensional parameter  $\theta$  on the diagonal is merely an *auxiliary* variable. Using the covariance of  $n \cdot \omega + \theta$ , all estimates in  $\theta$  will be transformed into estimates in  $\omega$  in the Newton scheme construction of  $u$  and  $\theta$  is always *fixed* at 0 in sect. 6. In particular,

$$A + \mathcal{O}(\delta^{p+1}) = (A + \mathcal{O}(\delta^{p+1}))(\omega, u, v)$$

is a Töplitz matrix *independent* of  $\theta$ .

### 5.1 The initial estimate in $\theta$ .

Let  $N = |\log \delta|^s$  ( $s > 1$ ) be as in Proposition 4.3 and  $T_N(\theta) = T_N(\theta; u^{(1)}, v^{(1)})$ . We have the following estimates.

**Lemma 5.1.** *Assume that  $u^{(0)} = \sum_{k=1}^b a_k e^{ij_k \cdot x} e^{-i(\sqrt{j_k^2+1})t}$  a solution to the linear wave equation (1.5) satisfies the non-degeneracy conditions (i, ii), and  $a \in \mathcal{B} \setminus \mathcal{B}_{\epsilon', \epsilon, \delta}$ , the open set defined in Proposition 4.3, with a further restriction to  $\epsilon \in (0, 1/2)$  (in view of (4.17, 4.18)). There exists  $\delta_0 > 0$ , such that for all  $\delta \in (0, \delta_0)$*

$$\|[T_N(\theta)]^{-1}\| \leq \delta^{-p-\epsilon} < e^{N^\sigma} \quad (5.2)$$

for some  $\sigma \in (0, 1)$  and there exists  $\beta \in (0, 1)$ , depending only on  $\text{supp } u^{(0)}, p, b, d$  and the  $H$  in (1.1), such that

$$|[T_N(\theta)]^{-1}(x, y)| \leq \delta^{\beta|x-y|} = e^{-\beta|\log \delta||x-y|} \quad (5.3)$$

for all  $x, y$  such that  $|x - y| > 1/\beta^2$ , for  $\theta$  away from a set  $B_N(\theta) \subset \mathbb{R}$  with

$$\text{meas } B_N(\theta) < \delta^{p+\epsilon/8b} < e^{-N^\tau}, \quad (5.4)$$

for some  $\tau \in (0, 1)$ .

The proof is related to that of Lemma 4.1 and rests on variable reduction. Note, however, that due to the presence of  $\theta$ , the third non-degeneracy condition cannot be used. The proof is divided into three steps.

#### *Eigenvalue spacing*

The new element in the proof, compared to that of Lemma 4.1, is the spacing of square roots. Toward that end define

$$\mathcal{W} := \mathcal{W}_N = \begin{pmatrix} \text{diag } (n \cdot \omega^{(0)} + \sqrt{j^2 + 1}) & 0 \\ 0 & \text{diag } (-n \cdot \omega^{(0)} + \sqrt{j^2 + 1}) \end{pmatrix} \quad (5.5)$$

to be the diagonal matrix restricted to  $[-N, N]^{b+d} \times \{0, 1\}$ . Denote by  $\lambda_i, \lambda_{i'}$  ( $i \neq i'$ ) two eigenvalues of  $\mathcal{W}$ . Then

$$\begin{aligned}\rho &:= \lambda_i \pm \lambda_{i'} \\ &= \nu \cdot \omega^{(0)} + \sqrt{j^2 + 1} \pm \sqrt{j'^2 + 1},\end{aligned}$$

for some  $\nu \in [-2N, 2N]^b$ ,  $j, j' \in [-N, N]^d$ . We have the following dichotomy:

(D1) If  $\rho = 0$ , then  $\nu$  has at most 2 non-zero components.

(D2) If  $\rho \neq 0$ , then

$$|\rho| \geq \frac{1}{|\log \delta|^{s\mathcal{L}}} \quad (\nu \neq 0),$$

for some  $\mathcal{L} > 1$ . (Recall that  $N = |\log \delta|^s$ ,  $s > 1$ .)

*Proof.* Assume that  $\nu$  has at least 3 non-zero components. Using the same square free reasoning leading to property (††) in sect. 2, proves that (D1) holds. For concreteness, take the  $\rho$  with the “−” sign, “+” sign works the same way. If  $\rho \neq 0$ , one may assume that

$$\begin{aligned}\rho_1 &= \nu \cdot \omega^{(0)} - (\sqrt{j^2 + 1} - \sqrt{j'^2 + 1}) \neq 0; \\ \rho_2 &= \nu \cdot \omega^{(0)} + (\sqrt{j^2 + 1} + \sqrt{j'^2 + 1}) \neq 0; \\ \rho_3 &= \nu \cdot \omega^{(0)} - (\sqrt{j^2 + 1} + \sqrt{j'^2 + 1}) \neq 0.\end{aligned}$$

As otherwise if  $\rho_i = 0$  for some  $i = 1, 2, 3$ , then

$$|\rho| \geq 2 \min(|\nu \cdot \omega^{(0)}|, 1) \geq \frac{1}{|\log \delta|^{s\mathcal{L}}} \quad (\nu \neq 0).$$

So (D2) is satisfied.

Multiplying  $\rho$  by  $\rho_1$ ,  $\rho_2$  by  $\rho_3$  produce

$$(\nu \cdot \omega^{(0)})^2 - (j^2 + j'^2 + 2) \neq \pm 2\sqrt{j^2 + 1}\sqrt{j'^2 + 1}.$$

Squaring yields

$$I := (\nu \cdot \omega^{(0)})^4 - 2(\nu \cdot \omega^{(0)})^2(j^2 + j'^2 + 2) + (j^2 - j'^2)^2 \neq 0. \quad (\dagger\dagger\dagger)$$

Rewrite the difference of the first two terms in  $I$  as

$$\sum_{k'=1}^{b'} C_{k'} w_{k'} + N',$$

35

where  $w_{k'} \neq 1$  are square roots of square free integers determined by the prime decompositions of  $(j_k^2 + 1)$ ,  $k = 1, 2, \dots, b$ ,  $C_{k'}, N' \in \mathbb{Z}$ . If all  $C_{k'} = 0$ , then since  $I \neq 0$ ,

$$|I| \geq 1;$$

otherwise using [Schm] and the restrictions on  $\nu, j, j'$  yields (D2).  $\square$

*Remark.* We note that unlike  $(\dagger\dagger)$ ,  $(\dagger\dagger\dagger)$  is weaker than a Diophantine property as it is only for certain rational combinations, but it suffices to establish Lemma 5.1.

*Size of connected sets on the bi-characteristics*

Let  $\sigma(\pm\mathcal{W})$  be the spectrum of  $\pm\mathcal{W}$ . For each  $\Theta \in \sigma(\mathcal{W}) \cup \sigma(-\mathcal{W})$ , define the bi-characteristics to be

$$\mathcal{C}_{\pm}(\Theta) = \{(n, j) \in \mathbb{Z}^{b+d} \mid \pm(n \cdot \omega^{(0)} + \Theta) - \sqrt{j^2 + 1} = 0\},$$

and

$$\mathcal{C}(\Theta) = \mathcal{C}_+(\Theta) \oplus \mathcal{C}_-(\Theta).$$

Define the connected sets on  $\mathcal{C}(\Theta)$  as in the paragraph containing (2.10), at the very beginning of sect. 2.2.

**Lemma 5.2.** *Assume that  $u^{(0)} = \sum_{k=1}^b a_k e^{ij_k \cdot x} e^{-i(\sqrt{j_k^2 + 1})t}$  satisfies the non-degeneracy conditions (i, ii). Then the connected sets  $\alpha$  on  $\mathcal{C}(\Theta)$  are at most of size  $4b$  for all  $\Theta \in \sigma(\mathcal{W}) \cup \sigma(-\mathcal{W})$ .*

*Proof.* We work separately on  $\mathcal{C}_+$  and  $\mathcal{C}_-$ . If there are two distinct points  $(n, j), (n', j') \in \mathcal{C}_+$ :

$$\begin{cases} (n \cdot \omega^{(0)} + \Theta) - \sqrt{j^2 + 1} = 0, \\ (n' \cdot \omega^{(0)} + \Theta) - \sqrt{j'^2 + 1} = 0, \end{cases}$$

and are, moreover, connected, then

$$(n' - n, j' - j) = (\nu, \eta) \in \Gamma$$

by definition, where  $\Gamma$  as defined in (2.6). Since  $\nu \neq 0$ , if  $\eta = 0$ , these two equations are incompatible using  $(\dagger)$ . One may therefore assume below that  $\eta \neq 0$ . Subtracting the second from the first and squaring lead to the following equation:

$$2j \cdot \eta + \eta^2 - (\nu \cdot \omega^{(0)})^2 + 2(\nu \cdot \omega^{(0)})\sqrt{j^2 + 1} = 0, \quad (**)$$

cf. (2.14). As before, call  $(n, j)$  the root.

More generally, if there is a connected set of  $(r + 1)$  sites on  $\mathcal{C}_+$ , choosing a root, there is a system of  $r$  equations of the form  $(**)$ , with  $(\nu, \eta) \in \tilde{\Gamma}$ , where  $\tilde{\Gamma}$  is as defined

in  $(\sharp)$ . Let  $S = \{(\nu_i, \eta_i)\}_{i=1}^r$  be the set of  $(\nu, \eta)$  appearing in  $(**)$ . From the dichotomy (D1), if

$$(\nu_i, \eta_i) \in S, i = 1, 2, \dots, r,$$

then  $\eta_i$  is a function of at most 2 variables in  $\{j_k\}_{k=1}^b$  and so are the difference functions

$$\eta_i - \eta_{i'}$$

for all  $i, i' = 1, 2, \dots, r, i \neq i'$ . Moreover one may assume that  $\eta_i \neq 0$  and  $\eta_i - \eta_{i'} \neq 0$  for all  $i, i', i \neq i'$ , as mentioned earlier. (Otherwise there is a contradiction to  $S$  being connected.) We prove the Lemma by contradiction as follows.

Assume that  $r = 2$  and that there is a constant  $C \neq 0$  such that  $\eta_2 \equiv C\eta_1, \nu_2 = C\nu_1$ . From (D1) and without loss of generality, one may assume that

$$0 \neq \eta_1 = mj_1 + hj_2.$$

Then

$$\eta_1^2 - (\nu_1 \cdot \omega^{(0)})^2 = -m^2 - h^2 + 2mh(j_1 \cdot j_2 - \sqrt{j_1^2 + 1}\sqrt{j_2^2 + 1}) \neq 0,$$

for  $j_1, j_2 \in \mathbb{Z}^d \setminus \{0\}$ . So the two equations of the form  $(**)$  are incompatible if  $C \neq 1$ . Therefore  $|\alpha| \leq 2$ .

Assume that  $r = 3$  and that there exist constants  $C_1 \neq 0, C_2 \neq 0$  such that

$$C_1\eta_1 + C_2\eta_2 + \eta_3 \equiv 0,$$

and there are *no* constants  $C'_i$ , such that

$$C'_i\eta_i \equiv \eta_{i'}, i, i' = 1, 2, 3, i \neq i'.$$

Similar to  $r = 2$ , one may assume that

$$0 \neq \eta_1 = m_1j_1 + h_1j_2,$$

$$0 \neq \eta_2 = m_2j_1 + h_2j_2,$$

$$0 \neq \eta_3 = m_3j_1 + h_3j_2.$$

If the system of the corresponding 3 equations of the form  $(**)$  is satisfied, then

$$C_1(\eta_1^2 - (\nu_1 \cdot \omega^{(0)})^2) + C_2(\eta_2^2 - (\nu_2 \cdot \omega^{(0)})^2) + (\eta_3^2 - (\nu_3 \cdot \omega^{(0)})^2) = 0,$$

where  $(\star)$  is used to deduce that

$$C_1\eta_1 + C_2\eta_2 + \eta_3 \equiv 0 \Leftrightarrow C_1\nu_1 + C_2\nu_2 + \nu_3 = 0,$$

cf. Lemma 2.1 in sect. 2.1 of [W1].

View  $C_1, C_2$  as the unknown and write in the  $j_1, j_2$  basis. In order for the 3 equations to be compatible, the determinant  $\mathcal{D}$  of the  $3 \times 3$  matrix must satisfy:

$$\mathcal{D} = \det \begin{pmatrix} m_1 & h_1 & \eta_1^2 - (\nu_1 \cdot \omega^{(0)})^2 \\ m_2 & h_2 & \eta_2^2 - (\nu_2 \cdot \omega^{(0)})^2 \\ m_3 & h_3 & \eta_3^2 - (\nu_3 \cdot \omega^{(0)})^2 \end{pmatrix} = 0.$$

Below we analyze the variety (in  $m_i, j_i, i = 1, 2, 3$ ) defined by  $\mathcal{D} = 0$ .

We first consider the case

$$m_i \neq 0, h_i \neq 0,$$

for all  $i = 1, 2, 3$ . Let

$$\begin{aligned} \tilde{\eta}_i &= \frac{1}{m_i h_i} \eta_i, \\ R_i &= \frac{1}{m_i h_i} (\eta_i^2 - (\nu_i \cdot \omega^{(0)})^2) \\ &= -\left(\frac{m_i}{h_i} + \frac{h_i}{m_i}\right) + 2(j_1 \cdot j_2 - \sqrt{j_1^2 + 1} \sqrt{j_2^2 + 1}), \end{aligned}$$

for  $i = 1, 2, 3$ . Without loss of generality, there are two cases:

- a)  $\tilde{\eta}_2 - \tilde{\eta}_1, \tilde{\eta}_3 - \tilde{\eta}_1$  span  $\mathbb{R}^2$ ;
- b)  $\tilde{\eta}_2 - \tilde{\eta}_1, \tilde{\eta}_3 - \tilde{\eta}_1$  are co-linear.

Case a) Using row reduction, the  $3 \times 3$  determinant (in the  $j_1, j_2$  basis as before)

$$\begin{aligned} D &= \det \begin{pmatrix} \tilde{\eta}_1 & R_1 \\ \tilde{\eta}_2 & R_2 \\ \tilde{\eta}_3 & R_3 \end{pmatrix} \\ &= P_1(j_1 \cdot j_2) - P_1 \sqrt{j_1^2 + 1} \sqrt{j_2^2 + 1} + P_2, \end{aligned}$$

where  $P_1$  and  $P_2$  are rational functions of  $m_i, h_i, i = 1, 2, 3$ , and

$$P_1 = 2 \det(\tilde{\eta}_2 - \tilde{\eta}_1, \tilde{\eta}_3 - \tilde{\eta}_1) \neq 0.$$

So  $D \neq 0$ , leading to a contradiction.

Case b) Since

$$\tilde{\eta}_3 - \tilde{\eta}_1 = C(\tilde{\eta}_2 - \tilde{\eta}_1), C \neq 0,$$

we have

$$\frac{\frac{1}{h_3} - \frac{1}{h_1}}{\frac{1}{h_2} - \frac{1}{h_1}} = \frac{\frac{1}{m_3} - \frac{1}{m_1}}{\frac{1}{m_2} - \frac{1}{m_1}} = C,$$

assuming  $m_1 \neq m_2$  and  $h_1 \neq h_2$ . Since

$$\begin{aligned} D &= P_2 \\ &= \left[ -\left(\frac{m_2}{h_2} + \frac{h_2}{m_2}\right) + \left(\frac{m_1}{h_1} + \frac{h_1}{m_1}\right) \right] \det(\tilde{\eta}_1, \tilde{\eta}_3 - \tilde{\eta}_1) \\ &\quad + \left[ \left(\frac{m_3}{h_3} + \frac{h_3}{m_3}\right) - \left(\frac{m_1}{h_1} + \frac{h_1}{m_1}\right) \right] \det(\tilde{\eta}_1, \tilde{\eta}_2 - \tilde{\eta}_1), \end{aligned}$$

if  $D = 0$ , there is the additional equality

$$\frac{\left(\frac{m_3}{h_3} + \frac{h_3}{m_3}\right) - \left(\frac{m_1}{h_1} + \frac{h_1}{m_1}\right)}{\left(\frac{m_2}{h_2} + \frac{h_2}{m_2}\right) - \left(\frac{m_1}{h_1} + \frac{h_1}{m_1}\right)} = C, \quad C \neq 0,$$

assuming  $\frac{m_1}{h_1} + \frac{h_1}{m_1} \neq \frac{m_2}{h_2} + \frac{h_2}{m_2}$ .

We view  $(m_1, h_1), (m_2, h_2)$  as the given and write

$$x = \frac{1}{m_3}, \quad y = \frac{1}{h_3}$$

as the unknown. We therefore have two equations of the form

$$\begin{cases} \frac{x-a_1}{a_2} = \frac{y-b_1}{b_2}, \\ \frac{x-a_1}{a_2} = \frac{(\frac{x}{y} + \frac{y}{x})-c_1}{c_2}, \end{cases}$$

where  $a_i \neq 0, b_i \neq 0, c_i \neq 0, i = 1, 2$  are the given. Solving for  $y$  from the first equation and substituting into the second lead to a (non-zero) polynomial equation in  $x \in \mathbb{R}$  of degree 3. So there are at most 3 solutions. Likewise, if  $b_2 = 0$ , then  $a_2 \neq 0$  and vice versa, and there are at most 2 solutions using similar arguments.

We now consider the remaining cases. Write

$$\mathcal{D} = \mathcal{P}_1(j_1 \cdot j_2) - \mathcal{P}_1 \sqrt{j_1^2 + 1} \sqrt{j_2^2 + 1} + \mathcal{P}_2,$$

where  $\mathcal{P}_1, \mathcal{P}_2 \in \mathbb{Z}$ , are functions of  $m_i, h_i, i = 1, 2, 3$ . If  $m_1 h_1 = 0$  (without loss of generality, one may assume  $m_1 = 0$ ), and  $m_2 h_2 \neq 0, m_3 h_3 \neq 0, \mathcal{P}_1 = 0$  if and only if  $h_3 = h_2$ . In that case  $\mathcal{D} = \mathcal{P}_2 = 0$  has at most 1 solution in  $m_3$ . Finally if  $m_1 h_1 = m_2 h_2 = 0$ , then  $\mathcal{D} \neq 0$ .

So in conclusion at most 5  $\eta_i$ 's of the form

$$\eta_i = m_i j_1 + h_i j_2$$

could possibly lead to compatible equations. Therefore

$$|S| = |\{(\nu_i, \eta_i)\}| \leq 5$$

and

$$|\alpha| \leq 6.$$

Lastly, assume that there is a connected set  $\alpha$  on  $\mathcal{C}_+$  with  $|\alpha| = 2b + 1$ . Then there are  $2b$  equations of the form (\*\*). From the restrictions on the variable dependence of  $\eta$ , one can always choose a root so that  $\eta$  in the set  $S$  contains a subset of  $b + 2$  elements of the forms

$$\begin{aligned} 0 \neq \eta_1 &= m'_1 j_1, \\ 0 \neq \eta_2 &= m_1 j_1 + h_2 j_2, \\ &\vdots \\ 0 \neq \eta_\ell &= m_1 j_1 + h_\ell j_\ell, \\ &\vdots \\ 0 \neq \eta_b &= m_1 j_1 + h_b j_b, \\ 0 \neq \eta_{b+1} &= m_1 j_1 + h_{b+1} j_\ell, \\ 0 \neq \eta_{b+2} &= m_1 j_1 + h_{b+2} j_\ell, \end{aligned}$$

for some  $\ell \in \{2, \dots, b\}$ . The arguments for  $r = 2, 3$  systems (when  $r = 3$ , the “remaining cases”) show that there is no solution to the subsystem formed by  $\eta_1, \eta_\ell, \eta_{b+1}, \eta_{b+2}$ . Therefore

$$|\alpha| \leq 2(b - 1) + 1 + 1 = 2b.$$

Multiplying by a factor of 2 to take into account  $\mathcal{C}_-$  proves the assertion.  $\square$

*Remark 1.* It is worth noting that the integer nature of  $j_k$ ,  $k = 1, 2, \dots, b$ , (aside from the non-degeneracy condition (ii)) is used in the variable reductions. This is contrary to the proof of the genericity of condition (iii) in the Lemma (sect. 2.1) and also that in sect. 2.2 of [W1].

*Remark 2.* Clearly specializing to  $\Theta = 0$ , this proves a weaker version of Proposition 2.1 with the bound  $4b$  instead of  $2d$ . The non-degeneracy condition (iii) is what permits the latter, in general, sharper bound on the bi-characteristics  $\mathcal{C}$  defined in (1.8).

*Proof of Lemma 5.1.* Let

$$\Theta \in \sigma(\mathcal{W}) \cup \sigma(-\mathcal{W}),$$

where  $\mathcal{W}$  defined as in (5.5). Since  $|n| \leq N = |\log \delta|^s$  ( $s > 1$ ),  $\omega = \omega^{(0)} + \mathcal{O}(\delta^p)$ ,  $\Delta u^{(1)} = \mathcal{O}(\delta^{2-\epsilon})$  and  $A(u^{(1)}) = \mathcal{O}(\delta^p)$ , in view of (5.1, 5.5) and (D2), for small  $\delta$ , it



suffices to look at  $\theta$  such that

$$\begin{aligned}\theta &\in \bigcup_{\Theta \in \sigma(\mathcal{W}) \cup \sigma(-\mathcal{W})} \{\Theta + \delta^p[-C|\log \delta|^s, C|\log \delta|^s]\} \\ &:= \bigcup_{\Theta \in \sigma(\mathcal{W}) \cup \sigma(-\mathcal{W})} \Theta + I,\end{aligned}$$

for some  $C > 1$ . This is because, otherwise  $T_N(\theta)$  is invertible, satisfying

$$\|T_N^{-1}(\theta)\| \leq \delta^{-p}.$$

Write  $\theta = \Theta + \delta^p \theta'$ . Let

$$w = \delta^{-1}a$$

and write

$$\omega = \omega^{(0)} + \delta^p \omega'(w).$$

We have

$$\begin{aligned}T_N(\theta) &= \begin{pmatrix} \text{diag} [\sqrt{j^2+1}(n \cdot \omega^{(0)} + \sqrt{j^2+1} + \Theta)] & 0 \\ 0 & \text{diag} [\sqrt{j^2+1}(-n \cdot \omega^{(0)} + \sqrt{j^2+1} - \Theta)] \end{pmatrix} \\ &\quad + \delta^p \begin{pmatrix} \text{diag} [\sqrt{j^2+1}(n \cdot \omega' + \theta')] & 0 \\ 0 & \text{diag} [\sqrt{j^2+1}(-n \cdot \omega' - \theta')] \end{pmatrix} + \delta^p A_N(u^{(0)}) \\ &\quad + \mathcal{O}(\delta^{p+1}),\end{aligned}$$

where  $A_N$  is the restricted  $A$  as defined in (2.5).

Let

$$\mathcal{K} = \begin{pmatrix} \text{diag} [\sqrt{j^2+1}(n \cdot \omega' + \theta')] & 0 \\ 0 & \text{diag} [\sqrt{j^2+1}(-n \cdot \omega' - \theta')] \end{pmatrix} + A_N$$

Let  $P_{\pm}$  be the projection onto  $\mathcal{C}_{\pm}(\Theta)$ . Define

$$P = \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix}$$

and  $P^c = I - P$ .

It follows from Lemma 5.2 that

$$PKP = \oplus_{\alpha} \mathcal{K}_{\alpha}(\theta', w), \tag{5.6}$$

where  $\theta'$  is in the size  $\mathcal{O}(|\log \delta|^s)$  interval  $I$  introduced earlier, and the direct sum is over connected sets on  $\mathcal{C}$ , and  $\mathcal{K}_{\alpha}$  is  $\mathcal{K}$  restricted to  $\alpha$  and therefore matrices of sizes at most  $4b \times 4b$  for all  $\alpha$ .

We proceed using the Schur reduction as in the proof of Lemmas 4.1. It suffices to estimate  $[PT_N P]^{-1}$ , as  $P^c T_N P^c$  is invertible using (D2), and

$$\|(P^c T_N P^c - \lambda)^{-1}\| \leq 4|\log \delta|^{s\mathcal{L}}$$

uniformly in  $\theta'$  for

$$\lambda \in \left[ -\frac{1}{4|\log \delta|^{s\mathcal{L}}}, \frac{1}{4|\log \delta|^{s\mathcal{L}}} \right].$$

Since  $T_N = \mathcal{K} + \mathcal{O}(\delta^{p+1})$ , this entails estimating

$$[PKP]^{-1} = \oplus_{\alpha} [\mathcal{K}_{\alpha}(\theta', w)]^{-1}.$$

The determinant of a block matrix  $\det \mathcal{K}_{\alpha}$  is a polynomial in  $\theta'$  of degree at most

$$M \leq 4b$$

with the coefficient in front of the highest degree term  $C_M$  satisfying

$$|C_M| > 1.$$

Consequently, variation in  $\theta'$  and summing over the number of  $\Theta$  in  $\sigma(\mathcal{W}) \cup \sigma(-\mathcal{W})$ , satisfying  $|\Theta| \leq \mathcal{O}(|\log \delta|^s)$  proves (5.2, 5.4). Afterwards the point-wise estimates in (5.3) follows. This is as in the proof of Lemma 4.1.  $\square$

## 5.2 Diophantine $\omega$ .

In Proposition 4.3, (4.23),  $\omega^{(1)}$  is shown to be Diophantine when restricted to the scale  $N = |\log \delta|^s, s > 1$ , using that  $\omega^{(0)}$  is Diophantine and small  $\delta$ . Below we give a general proof for unrestricted  $N$ .

**Lemma 5.3.** *Assume that  $\omega^{(0)}$  is a Diophantine vector in  $\mathbb{R}^b$  satisfying*

$$\|n \cdot \omega^{(0)}\|_{\mathbb{T}} \geq \frac{2\xi}{|n|^{\gamma}}, \quad n \in \mathbb{Z}^b \setminus \{0\}, \xi > 0, \gamma > 2b.$$

Let

$$\omega = \omega^{(0)} + \delta^p \omega',$$

where  $\mathcal{O}(1) = |\omega'| \leq 1$ , and  $\delta$  is small. Then  $\omega$  is Diophantine satisfying

$$\|n \cdot \omega\|_{\mathbb{T}} \geq \frac{\xi}{|n|^{3\gamma}}, \quad n \in \mathbb{Z}^b \setminus \{0\}, \xi > 0, \gamma > 2b, \quad (5.7)$$

away from a set in  $\omega'$  of measure less than  $\mathcal{O}(\delta^p)$ .

*Proof.* If

$$|n|\delta^p \leq \frac{\xi}{2|n|^\gamma},$$

clearly (5.7) holds. If

$$|n|\delta^p > \frac{\xi}{2|n|^\gamma},$$

then

$$|\delta^p n \cdot \omega' + n \cdot \omega^{(0)} + j| < \frac{\xi}{|n|^{3\gamma}}$$

for all  $j \in \mathbb{Z}$ , leads to a set in  $\omega'$  of measure less than  $\mathcal{O}(\delta^p)$ . This is because for each given  $n$ ,  $j$  may be restricted to  $|j| \sim |n|$ . Summing over  $(n, j) \in \mathbb{Z}^{b+1}$  then gives the measure estimate.  $\square$

*Remark.* Using Lemma 5.2 and Proposition 3.1,  $\omega^{(1)}$  satisfies the unrestricted Diophantine property (5.7) with a further excision in  $a$ .

### 5.3 The separation lemma.

For a fixed  $\theta$ , the region in  $\mathbb{Z}^{b+d}$  where  $D'(\theta)$  defined in (5.1) is small:

$$|D'(\theta; n, j; n, j)| < \mathcal{O}(\delta^p), \quad (5.8)$$

plays an important role. It is the generalization of the bi-characteristics hyperboloid  $\mathcal{C}$  defined in (1.8), sect. 1.1. To continue the iteration, it is essential that for all  $\theta$ , the subset in  $\mathbb{Z}^{b+d}$  defined by (5.8) has the following separation property.

**Lemma 5.4.** *Let  $B$  be a large integer and assume that  $\omega \in \mathbb{R}^b$  satisfies that*

$$|P(\omega)| > \delta^p B^{-C'}, \quad (5.9)$$

*for all non-zero even quadratic polynomials*

$$0 \neq P(\chi) \in \{[-\mathcal{O}(B^{2d}), \mathcal{O}(B^{2d})] \cap \mathbb{Z}\}[\chi_1, \chi_2, \dots, \chi_b]. \quad (5.10)$$

*Consider a sequence, a subset  $\{x_i\}_{i=1}^\ell \subset \mathbb{Z}^{b+d}$  satisfying*

$$|x_i - x_{i-1}| < B,$$

*such that for some  $\theta \in \mathbb{R}$  and for all  $i$*

$$|(n_i \cdot \omega + \theta)^2 - j_i^2 - 1| < \mathcal{O}(\delta^p), \quad (5.11)$$

where  $(n_i, j_i) = x_i$ . Let  $\pi$  be the projection onto  $\mathbb{Z}^d$ . Assume, moreover that for all  $j_i$ , the number of  $x_i$  such that

$$\pi x_i = j_i$$

is less than  $B'$ . Then

$$\ell < (BB')^{C''}. \quad (5.12)$$

(The order  $\mathcal{O}$  and the constant  $C''$  depend on  $C'$ ,  $d$  and  $b$ .)

*Proof.* After the rescaling:  $(n, j) \rightarrow \delta^{-p/2}(n, j)$ ; more precisely, after (re)defining the operators

$$T_{\pm} : \mathbb{R}^{b+d} \mapsto \mathbb{R}^{1+d}$$

to be

$$(n, j) \mapsto \delta^{-p/2}(\pm n \cdot \omega, j),$$

the proof becomes the same as that of Lemma 20.14 in Chap. 20 of [B4]. Moreover even quadratic polynomial conditions on  $\omega$  suffice for the proof. This can be deduced as follows.

Let

$$A_{ii'}^{\pm} = (n_i \cdot \omega)(n_{i'} \cdot \omega) \pm j_i \cdot j_{i'},$$

where  $1 \leq i, i' \leq d_1 \leq d$ , and  $A^{\pm}$  the corresponding  $d_1 \times d_1$  matrices. The sole purpose of the nonlinear polynomial conditions on  $\omega$  in [B4] is to ensure an appropriate lower bound on  $|\det A^{\pm}|$ . Since

$$\det A^{\pm} = \wedge_{i=1}^{d_1} [(n_i \cdot \omega) \begin{pmatrix} n_1 \cdot \omega \\ n_2 \cdot \omega \\ \vdots \\ n_{d_1} \cdot \omega \end{pmatrix} \pm \begin{pmatrix} j_i \cdot j_1 \\ j_i \cdot j_2 \\ \vdots \\ j_i \cdot j_{d_1} \end{pmatrix}],$$

using the antisymmetry of the wedge product and that the first vectors in the square brackets are all co-linear, this actually only leads to even quadratic polynomials in  $\omega$  with integer coefficients. So (5.9, 5.10) suffice. (Cf. also [BeBo].)

The  $\mathcal{O}(\delta^p)$  upper bound in (5.8, 5.9) is in consideration of the measure estimates in  $a$ , as for later applications  $\|(\partial\omega/\partial a)^{-1}\| \lesssim \delta^{-p+1}$  only, cf. (4.21).  $\square$

#### 5.4 The iteration lemmas in $\theta$ .

Let  $T(\theta)$  be defined as in (5.1), with in addition,  $u$  standing for  $u^{(1)}$ . The purpose of this subsection is to establish the analogues of Lemma 5.1 for all scales inductively. The proof essentially follows and adapts that of Lemmas 19.13, 20.19, 19.38 and 19.56 in Chaps. 19 and 20 of [B4].

We first restrict to an interval  $I \subset \mathbb{R}^{2b}$  and assume that  $u = u(a, \omega)$  is holomorphic in  $a, \omega$  on  $I$  and that

$$\|\partial_a u\| < C, \|\partial_\omega u\| < C, |u(x)| < e^{-|x|^c}, 0 < c < 1. \quad (5.13)$$

(These assumptions will be verified along the iteration process, cf. sect. 6.)

The invertibility analysis of the linearized operator  $T(\theta)$  is multi-scale and consists of non-perturbative as well as perturbative aspects. The cubes with “small”  $j$  coordinates, are dealt with algebraically: the invertibility of the restricted  $T(\theta)$  is obtained by making a semi-algebraic description of the singular sets and studying their complexity; while eigenvalue variation arguments and Lemma 5.4 are used to control the cubes with large  $|j|$ .

#### *The perturbative region*

We first consider cubes of the form:

$$\Lambda_{N,J} := [-N, N]^{b+d} + (0, J), \quad (5.14)$$

where

$$\|J\|_\infty > \text{const.} N,$$

for some  $\text{const.} > 1$ . Denote by  $T_{N,J}$  the linearized operator restricted to the cube  $\Lambda_{N,J}$ . Invertibility in this (elliptic) region is obtained by appealing to eigenvalue variations as in the proof of Lemmas 19.13, 20.19 in [B4]. Below we reproduce the arguments.

**Lemma 5.5.** *Assume that  $\omega$  is Diophantine, satisfying (5.7) for  $|n| \leq N$ , as well as the even quadratic conditions in (5.9-5.10) with  $N$  in place of  $B$ . On  $I \subset \mathbb{R}^{2b}$ , there are  $C_1, C' > 1$  and a system of Lipschitz functions*

$$\theta_i = \theta_i(a, \omega), 1 \leq i < e^{(\log N)^{C_1}},$$

*satisfying*

$$\|\theta_i\|_{\text{Lip}} < C' N.$$

*There exist  $0 < \kappa < \sigma < c < 1$ , so that the following holds:*

$$\|T_{N,J}^{-1}(\theta)\| < e^{N^\sigma}, \quad (5.15)$$

$$|T_{N,J}^{-1}(x, x')| < e^{-|x-x'|^c}, \quad (5.16)$$

*for  $|x - x'| > N/10$ , if*

$$\min_i |\theta - \theta_i(a, \omega)| > e^{-N^\kappa}, \quad (5.17)$$

for all scales  $N \geq |\log \delta|^s \gg 1$ ,  $s > 1$ .

Assume that the Lemma holds at scale  $N$ . The proof uses a multi-scale analysis from scale  $N$  to a larger scale  $N_1$  satisfying

$$N_1 = e^{N^{\kappa/2}}, \kappa > 0$$

cf. the proofs of Lemmas 20.19 and 19.13 in [B4]. It rests on eigenvalue variation and the following decomposition.

**Lemma 5.6.** *There exist  $0 < \sigma < c < 1$ ,  $C > 1$ ,  $0 < \rho < 1/C$  such that for any fixed  $\theta$ , there is a partition of  $\Lambda_{N_1, J}$ , defined as in (5.14) with  $N_1$  replacing  $N$ , into the good set  $\Omega_0$  and the bad set  $\Omega_1$ :*

$$\Lambda_{N_1, J} = \Omega_0 \cup \Omega_1.$$

*On the good set, the restricted matrix satisfies*

$$\|(T_{\Omega_0})^{-1}\| \lesssim e^{N^\sigma}, \quad (5.18)$$

*and the point-wise estimates*

$$|(T_{\Omega_0})^{-1}(x, x')| \lesssim e^{-|x-x'|^c}, \quad (5.19)$$

*for all  $x, x'$ , such that  $|x - x'| > 10N$ .*

*On the bad set there is a partition:*

$$\Omega_1 = \bigcup_{\beta} \Omega_{1, \beta},$$

*such that*

$$\text{diam } \Omega_{1, \beta} < N_1^{\rho C},$$

*and*

$$\text{dist } (\Omega_{1, \beta}, \Omega_{1, \beta'}) > N_1^\rho,$$

*for  $\beta \neq \beta'$ .*

*Proof.* Write  $\Lambda_{N_1}$  for  $\Lambda_{N_1, J}$ . Call a site  $x \in \Lambda_{N_1}$  good, if the estimates (5.15, 5.16) hold for  $T_{N, x}$ , the restricted operator on

$$\Lambda_x = [-N, N]^{b+d} + x.$$

Write

$$\Lambda_{N_1} = \Omega_0 \cup \Omega_1,$$

where  $\Omega_0$  is the good subset, if  $x \in \Omega_0$ , then  $T_{N,x}^{-1}$  satisfies (5.15, 5.16). Using a basic resolvent expansion, cf. Lemma 2.2 in [BGS],  $T_{\Omega_0}$  satisfies (5.18, 5.19).

If  $x \in \Omega_1$ , then there must be  $y \in \Lambda_{N_1}$  satisfying

$$|y - x| \lesssim N$$

such that

$$|(n \cdot \omega + \theta)^2 - j^2 - 1| < \mathcal{O}(\delta^p),$$

where  $(n, j) = y$ . Let

$$B = N_1^\rho, \tag{5.20}$$

with  $0 < \rho < 1/C < 1$ ,  $C$  will be determined at the end of the proof. Assume that there is a sequence of distinct elements  $x_k \in \Omega_1$  such that

$$|x_{k+1} - x_k| < B. \tag{5.21}$$

Denote by  $\Omega_{1,\beta}$ , this sequence. We may then find a corresponding sequence of distinct elements  $y_k$  such that

$$|y_{k+1} - y_k| < B + \mathcal{O}(N) < 2B$$

and

$$|(n_k \cdot \omega + \theta)^2 - j_k^2 - 1| < \mathcal{O}(\delta^p),$$

where  $(n_k, j_k) = y_k$ .

Write  $y_k = (n, j)$  and let  $z = (m, j)$ . To apply Lemma 5.4 to the sequence  $\{y_k\}$ , we need to bound  $M$ , the number of  $z$ , such that for a fixed  $\theta$ ,  $T_{z,N}$ , the operator restricted to  $z + [-N, N]^{b+d}$  is bad, violating (5.15, 5.16). Since the estimates (5.15, 5.16) are valid at scale  $N$ , (5.17) and Diophantine  $\omega$  satisfying (5.7) for  $|m| \leq N_1$  gives

$$M < e^{(\log N)^{C_1}}.$$

So

$$B' \lesssim N^d e^{(\log N)^{C_1}}.$$

Lemma 5.4 then gives that the number of elements in  $\{y_k\}$ ,

$$|\{y_k\}| < (BB')^{C''}.$$

So

$$\text{diam } \Omega_{1,\beta} < N_1^{\rho C}$$

with  $C \gtrsim C''$ , where we used (5.20). Combined with (5.20), this concludes the proof.

□

*Proof of Lemma 5.5.* The strategy of the proof is based on the geometric resolvent expansion, cf. Lemma 7 in the Appendix of [B3], using the decomposition in the proceeding lemma. Since the estimates on  $\Omega_0$  permit a  $N_1^{-2}e^{-N}$  perturbation in  $(a, \omega, \theta)$ , the decomposition remains valid in a neighbourhood of this size about a given initial point  $(\bar{a}, \bar{\omega}, \bar{\theta}) \in \mathbb{R}^{2b+1}$ . To apply the resolvent expansion, we only need estimates on

$$\|(T_{\tilde{\Omega}_{1,\beta}})^{-1}\|,$$

for all  $\beta$ , where

$$\tilde{\Omega}_{1,\beta} \supset \Omega_{1,\beta}.$$

is a neighborhood of  $\Omega_{1,\beta}$  of size  $N_1^{\rho/2}$ .

Since

$$|\sqrt{j^2 + 1}| \pm n \cdot \omega + \sqrt{j^2 + 1}| > |j|(|j| - \mathcal{O}(N)) \sim j^2,$$

for  $(n, j) \in \tilde{\Omega}_{1,\beta}$ , we use eigenvalue variations. Recall that  $\tilde{\Omega}_{1,\beta}$  denotes

$$\tilde{\Omega}_{1,\beta} \times \{0, 1\} \sim \{\tilde{\Omega}_{1,\beta}, 0\} \oplus \{\tilde{\Omega}_{1,\beta}, 1\} := \{\tilde{\Omega}_{1,\beta}, +\} \oplus \{\tilde{\Omega}_{1,\beta}, -\}$$

Without loss of generality, one may assume that  $\theta \geq 0$ , and that

$$\theta \in [J - \mathcal{O}(N), J + \mathcal{O}(N)],$$

as otherwise  $T_{N,J}^{-1}(\theta)$  is a priori invertible. So diagonals on  $\{\mathbb{Z}^{b+d}, +\}$ , satisfy

$$D(\theta) = \sqrt{j^2 + 1}|n \cdot \omega + \theta + \sqrt{j^2 + 1}| > |j|(|j| - \mathcal{O}(N)) \sim N_1^2 \gg 1,$$

and are nonsingular.

Let  $R_{\pm}$  be the projections onto  $\{\mathbb{Z}^{b+d}, \pm\}$ , then

$$\|(R_+ T_{\tilde{\Omega}_{1,\beta}} R_+)^{-1}\| \lesssim N_1^{-2}$$

and is invertible. The Schur complement reduction formula then reduces the estimate of

$$\|(T_{\tilde{\Omega}_{1,\beta}})^{-1}\|$$

to that of the eigenvalues of the *self-adjoint* effective matrix:

$$\begin{aligned} H &= R_- T_{\tilde{\Omega}_{1,\beta}} R_- + R_- T_{\tilde{\Omega}_{1,\beta}} R_+ (R_+ T_{\tilde{\Omega}_{1,\beta}} R_+)^{-1} R_+ T_{\tilde{\Omega}_{1,\beta}} R_- \\ &:= \theta + T'(a, \omega, \theta). \end{aligned}$$

The matrix  $H$  has nonlinear dependence on  $\theta$ . The important thing to notice is that  $R_- T_{\tilde{\Omega}_{1,\beta}} R_+$  only contains off-diagonal elements and is therefore of order  $\mathcal{O}(\delta^p)$ . This



also holds for its derivative with respect to  $\theta$ . Similarly this is true for  $R_+T_{\tilde{\Omega}_{1,\beta}}R_-$  and its derivative. This then yields

$$\|\partial_\theta T'\| < C\delta^{2p}\|(R_+T_{\tilde{\Omega}_{1,\beta}}R_+)^{-1}\|^2 < C\delta^{2p}N_1^{-4}.$$

So

$$\frac{\partial H}{\partial \theta} > \frac{1}{2}I$$

as an operator, where  $I$  is the identity matrix.

Clearly we also have the regularity estimates:

$$\|\partial_\omega T'\| \lesssim \delta^{p-1}\|\partial_\omega u\| + \mathcal{O}(N_1) < CN_1,$$

as well as

$$\|\partial_a T'\| < \|\partial_a u\| < C.$$

Self-adjointness of  $H$  and the positivity of  $\partial H/\partial \theta$  then permit eigenvalue variation estimates.

More precisely, let  $\{E_i(T')\}$  be the set of eigenvalues of  $T'$  arranged in ascending order. Then  $E_i$  are continuous functions of the parameters  $a, \omega, \theta$  and (from the analyticity assumption on  $u = u(a, \omega)$ ) separately, piece-wise holomorphic in each 1 dimensional parameter.

From first order eigenvalue variation, it follows that the eigenvalues  $E_i$  are Lipschitz satisfying

$$\|E_i\|_{\text{Lip}(a)} < C, \|E_i\|_{\text{Lip}(\omega)} < N_1, \|E_i\|_{\text{Lip}(\theta)} < N_1^{-4}.$$

From the definition of  $H$ , the set

$$\{\theta + T'(a, \omega, \theta)\}$$

is a parametrization of the spectrum of  $H$ ,  $\sigma(H)$ . Fix  $i$ , using the above Lipschitz estimates on  $E_i$ , the equation

$$\theta + E_i(a, \omega, \theta) = 0$$

defines a function

$$\theta = \theta_i(a, \omega).$$

Thus

$$\theta_i(a, \omega) + E_i(a, \omega, \theta_i(a, \omega)) = 0,$$

and from the Lipschitz estimates

$$|\theta_i(a, \omega) - \theta_i(a', \omega')| \leq N_1^{-4}|\theta_i(a, \omega) - \theta_i(a', \omega')| + C|a - a'| + N_1|\omega - \omega'|$$

implying in turn

$$\|\theta_i\|_{\text{Lip}(a)} \leq C, \|\theta_i\|_{\text{Lip}(\omega)} \leq N_1,$$

and

$$\begin{aligned} |\theta + E_i(a, \omega, \theta)| &= |\theta - \theta_i(a, \omega) + E_i(a, \omega, \theta) - E_i(a, \omega, \theta_i(a, \omega))| \\ &= |\theta - \theta_i(a, \omega)|(1 + \mathcal{O}(N_1^{-4})). \end{aligned}$$

Consequently,

$$\text{dist}(\sigma(H), 0) \sim \min_i |\theta - \theta_i(a, \omega)|.$$

Hence

$$\|(T_{\tilde{\Omega}_{1,\beta}})^{-1}\| \leq \max_i |\theta - \theta_i(a, \omega)|^{-1}.$$

Clearly  $\theta < 0$  can be dealt with in the same way. Collecting the functions  $\{\theta_i\}$  over all  $\beta$ , we obtain, at most of the order,  $N_1^{b+d}$  Lipschitz functions  $\{\theta_i = \theta_i(a, \omega)\}$  such that

$$\|\theta_i\|_{\text{Lip}} \lesssim N_1$$

and

$$\max_{\beta} \|(T_{\tilde{\Omega}_{1,\beta}})^{-1}\| \leq \max_i |\theta - \theta_i(a, \omega)|^{-1}.$$

Requiring

$$\max_i |\theta - \theta_i(a, \omega)|^{-1} < e^{N_1^\kappa}$$

with  $0 < \kappa < \rho c$ , then yields

$$\|(T_{\tilde{\Omega}_{1,\beta}})^{-1}\| \lesssim e^{N_1^\kappa},$$

for all  $\beta$ .

The above construction of the Lipschitz family of  $\theta_i$  is after restricting to a  $N_1^{-2}e^{-N}$  neighborhood of the initial parameter  $(\bar{a}, \bar{\omega}, \bar{\theta}) \in \mathbb{R}^{2b+1}$ . Clearly the order of the number of these neighborhoods is bounded by

$$\ell = N_1^{d+1}(N_1^2 e^N)^{2b+1},$$

where the additional factor  $N_1^d$  takes into account the nonlinear dependence of  $H$  on  $\theta$ . The total number of  $\theta_i$  functions is therefore at most of the order

$$N_1^{b+d}\ell < e^{(\log N_1)^{C_1}},$$

for some  $C_1 > 1$ .

Subsequently, the estimates on  $T_{N_1,j}^{-1}$  for  $\theta$  satisfying (5.17) follows from the geometric resolvent expansion provided

$$c = \frac{1 - \rho C}{1 - \rho} > 0$$

and the  $\rho$ ,  $C$  are as in Lemma 5.6.

Finally note that for the induction at scale  $N_1 = |\log \delta|^s$ , we need (5.15-5.17) at appropriate smaller scales  $N < |\log \delta|^s$ , which can be obtained by repeating the earlier eigenvalue variation on the entire  $\Lambda_{N,J}$ . This concludes the proof.  $\square$

### *The non-perturbative region*

We are now ready to state the estimates when  $T$  is restricted to cubes centered at the origin:

$$\Lambda_N = [-N, N]^{b+d},$$

or more generally, cubes

$$\Lambda_{N,J} = [-N, N]^{b+d} + (0, J),$$

with  $\|J\|_\infty \leq \text{const.} N$ , for some  $\text{const.} > 1$ . This is the more difficult, non-elliptic region. It uses the Cartan estimate in Proposition 14.1 in [B4], cf. [BGS], in addition to eigenvalue variations. It adapts and is akin to the proof of Lemma 19.38-19.56 in [B4].

Toward that end, let

$$f : (0, 1)^b \mapsto (0, 1)^b$$

be a diffeomorphism satisfying

$$\|(\partial f / \partial y)^{-1}\| \asymp O(1).$$

Then a set  $S \subset (0, 1)^{2b}$  is said to have sectional measure at most  $\mu$ :

$$\text{meas}_{\text{sec}} S < \mu,$$

if

$$\text{meas} \{y \in (0, 1)^b | (y, f(y)) \in S\} < \mu.$$

Let  $K$  be such that

$$\bar{\omega}_k = \frac{\omega_k - \omega_k^{(0)}}{\delta^p K} \in (0, 1)$$

for  $k = 1, 2, \dots, b$ , and  $w = \delta^{-1}a$ . Denote by  $I' \ni (w, \bar{\omega})$ , the image of the interval  $I$ , fixed at the beginning of this subsection. Let  $T_N$  be the operator restricted to the cube  $\Lambda_{N,J}$ . There are the following estimates.

**Lemma 5.7.** *Given  $N \in \mathbb{N}$ , assume that the conditions on  $\omega$  in Lemma 5.5 are satisfied and that  $\bar{\omega} = \bar{\omega}(w)$  satisfies  $\|(\partial\bar{\omega}/\partial w)^{-1}\| \asymp \mathcal{O}(1)$ . There exist  $0 < \tau < \sigma < c < 1$  such that for all  $N \geq |\log \delta|^s \gg 1$ ,  $1 < 1/\sigma \leq s \leq 1/\tau$ , there is a subset  $\mathcal{A} \subset I' \subset [0, 1]^{2b}$  in  $(w, \bar{\omega})$ , which is a union of intervals of size*

$$[\exp \exp(\log \log N)^3]^{-1},$$

and

$$\text{meas}_{\text{sec}} \mathcal{A} < [\exp \exp(\log \log N)^2]^{-1}.$$

If  $(w, \bar{\omega}) \in I' \setminus \mathcal{A}$ , then

$$\|T_N^{-1}(\theta)\| < e^{N^\sigma} \quad (5.22)$$

and

$$|T_N^{-1}(x, x')| < e^{-|x-x'|^c}, \quad (5.23)$$

for all  $x, x'$  such that  $|x - x'| > N/10$ , and  $\theta$  in the complement of a set  $\mathcal{B}_N(\theta)$  of measure

$$\text{meas } \mathcal{B}_N(\theta) < e^{-N^\tau}. \quad (5.24)$$

*Proof.* Let  $N_1 > N$  satisfy

$$\exp(\log \log N_1)^3 = N.$$

For concreteness, take the cube centered at the origin

$$\Lambda_{N_1} := [-N_1, N_1]^{b+d}.$$

(The proof is the same for  $J \neq 0$ .) The proof uses induction. The estimates needed in the induction are provided by Lemma 5.1 and its straight forward analogues for appropriate smaller scales.

Using the multi-scale reasoning, the induction from scale  $N$  to a larger scale  $N_1$  hinges on controlling the number of singular boxes at scale  $N$ . The subset

$$T_1 = \{(n, j) \mid \|j\|_\infty \leq \text{const.} N\} \cap \Lambda_{N_1},$$

is the non-perturbative region at the smaller scale  $N$ . It is covered by boxes

$$\Lambda_{N,j}^n = [-N, N]^{b+d} + (n, j),$$

with  $\|j\|_\infty \leq \text{const.} N$ . This is the region where eigenvalue variation is not applicable and instead, Cartan type of theorem or related arguments in [BGS] are used. The key is to ensure that the number of singular boxes are *sub-linear*.

We first make an additional excision of the frequency  $\omega$  in the perturbative region (at scale  $N$ ). The aim is to localize further the singular cubes and thereby reduce their total number. If a cube  $\Lambda_{N,j}^n$  in the perturbative region:

$$[-N, N]^{b+d} + (n, j), \quad (5.25)$$

where  $\|n\|_\infty \leq N_1$  and  $\|j\|_\infty > \text{const.} \cdot N$ , is singular, then there exists  $\theta_i$  in the family of Lipschitz functions as introduced in Lemma 5.5, such that

$$|\theta + n \cdot \omega - \theta_i| < e^{-N^\kappa}, \quad \kappa > 0. \quad (5.26)$$

We begin by localizing the singular cubes in the  $n$ -direction. We define the complement of  $\mathcal{A}$ :  $I' \setminus \mathcal{A}$  to be the set such that for

$$(w, \bar{\omega}) \in I' \setminus \mathcal{A}, \quad |n \cdot \omega - \theta_i(a, \omega) + \theta_{i'}(a, \omega)| > 2e^{-N^\kappa}, \quad (5.27)$$

for all  $n$  such that

$$N^{1+\bar{\epsilon}} < |n| \leq 2N_1, \quad \bar{\epsilon} > 0, \quad (5.28)$$

and for all  $\theta_i$  and  $\theta_{i'}$  in the Lipschitz family at scale  $N$ . (The choice of  $N^{1+\bar{\epsilon}} \gg N$  is in view of the Lipschitz estimates on  $\theta_i$  in Lemma 5.5.)

Let

$$d := \frac{1}{\|n\|} \sum_{k=1}^b n_k \partial_{\omega_k},$$

then

$$d(n \cdot \omega) = \|n\| > N^{1+\bar{\epsilon}} \gg N.$$

Since  $\omega$  and  $a$  are diffeomorphic, using the Lipschitz property of  $\theta_i$  in Lemma 5.5-Corollary 5.7 and that there are at most

$$N_1^b [N_1^d e^{(\log N)^{C_1}}]^2 < N_1^{b+2d+2}$$

conditions in (5.27) produces the measure estimates. Moreover the estimates in (5.27) are stable under perturbations of size  $e^{-N}$  in  $(w, \bar{\omega})$ , thereby generating a collection of intervals of the size stated above (5.22).

Here to compensate for the  $\|(\frac{\partial \omega}{\partial a})^{-1}\| \lesssim \delta^{-p+1}$  factor, cf. (4.21), we used that when the scales

$$N \leq |\log \delta|^s, \quad s > 1,$$

the right side of (5.27)  $e^{-N^\kappa}$  can be replaced by  $\delta^{p+\tilde{\epsilon}}$  with  $\tilde{\epsilon} \in (0, \epsilon)$  in view of (5.2) in Lemma 5.1 and the proof of Lemma 5.5.

If (5.27) holds, then there is  $\bar{n}$ , such that all singular cubes of the form in (5.25) satisfy

$$|n - \bar{n}| < N^{1+\bar{\epsilon}}. \quad (5.29)$$

If a cube  $\Lambda_{N,j}^n$  is singular, then necessarily there is  $(n', j') \in \Lambda_{N,j}^n$  such that

$$|(n' \cdot \omega)^2 - j'^2 - 1| < \mathcal{O}(\delta^p). \quad (5.30)$$

Since  $|n - n'| \leq N$ , combining (5.29) with (5.30) gives that there is a “cylindrical annulus”

$$A \subset \mathbb{Z}^{b+d}$$

of radius  $R$  and “width”  $N^{1+\bar{\epsilon}}$  in the  $\mathbb{Z}^d$ -direction and “height”  $N^{1+\bar{\epsilon}}$  in the  $\mathbb{Z}^b$  direction, such that if  $\Lambda_{N,j}^n$  is singular, then

$$\Lambda_{N,j}^n \cap A \neq \emptyset.$$

This localizes the singular cubes in the perturbative region:  $\Lambda_{N_1} \setminus T_1$  and yields that there are at most of the order

$$(\bar{N}^{1+\bar{\epsilon}})^{b+d} R^{d-1}$$

bad boxes.

To bound the number of singular boxes in the non-perturbative region, we use the Diophantine property (5.7), the measure estimate in (5.24) at scale  $N$  and that the sets  $\mathcal{B}_N$  are semi-algebraic, cf. Lemma 4.4 in [BW] and its proof. This gives that there are at most  $N^C$ ,  $C > 1$ , bad  $N$ -boxes in  $T_1$ . Adding the two estimates then produces the bound:

$$N^C + (N^{1+\bar{\epsilon}})^{b+d} R^{d-1} \lesssim N^C R^{d-1}, \quad C > b + d,$$

on the number of possibly singular cubes, cf. the proof of Lemma 19.38 and the end of that of Lemma 20.19 in [B4]. This leads to two cases:

- (i)  $R \leq N_1^{\sigma'/d}$ ,  $0 < \sigma' < \sigma$ ;
- (ii)  $R > N_1^{\sigma'/d}$ ,  $0 < \sigma' < \sigma$ .

(i) The total number of singular cubes are of the order  $N_1^{\sigma'}$ . (For  $d = 1$ , there are at most  $N^C$  bad boxes. Since  $N^C \ll N_1^{\sigma'}$ , this is the only possibility.) So the proof of Lemma 4.4 (Corollary 4.5) of [BGS] provide the estimates in (5.22-5.24) with

$$0 < \tau < \sigma - \sigma'.$$

- (ii) Since  $R$  is large, the non-perturbative  $T_1$  region may be enlarged to

$$\mathcal{T}_1 = \{(n, j) \mid \|j\|_\infty < N_1^{\sigma'/d}\} \cap \Lambda_{N_1} \supset T_1.$$

On  $\mathcal{T}_1$ , the singular  $N$  cubes are *all* in  $T_1$ , the total number is therefore of the order  $N^C \ll N_1^{\sigma'/d} \ll N_1^{\sigma'}$ . The proof of Lemma 4.4 (Corollary 4.5) of [BGS] are again applicable providing the norm estimate:

$$\|T_{\mathcal{T}_1}^{-1}\| < e^{N_1^{c\sigma'/2d}} \ll e^{N_1^\sigma}, \quad (5.31)$$

as well as the point-wise estimate

$$|T_{\mathcal{T}_1}^{-1}(x, x')| < e^{-|x-x'|^c}, \quad (5.32)$$

for all  $x, x' \in \mathcal{T}_1$  satisfying

$$|x - x'| > \frac{N_1^{\sigma'/d}}{10},$$

away from a set in  $\theta$  of measure

$$(e^{-N_1^{c\sigma'/2d}})^{1/N^C} \ll e^{-N_1^\tau},$$

if

$$0 < c\sigma'/2d < \tau.$$

Define

$$\mathcal{T}_2 = \{(n, j) \mid \|j\|_\infty > N_1^{\sigma''/d}\} \cap \Lambda_{N_1},$$

with  $0 < \sigma'' < \sigma'$ ;

$$\mathcal{T}_2 \cup \mathcal{T}_1 = \Lambda_{N_1}.$$

On  $\mathcal{T}_2$  we use perturbative arguments. A new scale  $N_2$  is introduced,

$$N_2 = N_1^{\sigma'''/d}$$

with

$$0 < \sigma''' < \sigma'' < \sigma',$$

so that Lemma 5.5 is available at scale  $N_2$ . Cover  $\mathcal{T}_2$  by the  $N_2$  cubes and restrict  $\theta$  so that all  $N_2$  cubes are good satisfying (5.15, 5.16). Since there are at most

$$N_1^{b+d} e^{(\log N_2)^{C_1}}$$

Lipschitz functions  $\theta_i$  for the  $N_2$  cubes from Lemma 5.5, this amounts to excluding a set in  $\theta$  of measure

$$N_1^{b+d} e^{(\log N_2)^{C_1}} e^{-N_2^\kappa} < e^{-N_1^\tau}$$

by choosing  $\tau, \sigma', \sigma'''$  such that

$$0 < c\sigma'/2d < \tau = \sigma''' \kappa / 2d < \sigma - \sigma'.$$

Call the set of  $\theta$ , on which (5.31, 5.32) hold for  $\mathcal{T}_1$  and (5.15, 5.16) hold on *all*  $N_2$  cubes in  $\mathcal{T}_2$ , the good set. On the good set, the basic resolvent expansion, cf. Lemma 2.2 of [BGS], using the  $N_2$  cubes and  $\mathcal{T}_1$  then gives (5.22-5.23). Summing over the measure estimates in  $\theta$  of the complement, the bad set, yields (5.24). Setting  $N = |\log \delta|^s$ , formulae (5.22, 5.24) then fix  $s$  to be

$$1 \leq 1/\sigma \leq s \leq 1/\tau. \quad (5.33)$$

This concludes the proof. (Cf. the proof of Lemma 19.38-19.65 in [B4]. )  $\square$

## 6. Proof of the Theorem

The proof of the Theorem is an induction. Let

$$w = \delta^{-1}a \in (0, 1]^b, \omega \in \delta^p(-K, K)^b + (\sqrt{j_1^2 + 1}, \sqrt{j_2^2 + 1}, \dots, \sqrt{j_b^2 + 1}),$$

where  $K = K(p, b)$  and define

$$\tilde{\omega} = [\omega - (\sqrt{j_1^2 + 1}, \sqrt{j_2^2 + 1}, \dots, \sqrt{j_b^2 + 1})/\delta^p \in (-K, K)^b.$$

Let  $M$  be a large integer. The proof consists in showing that on the *entire*  $(w, \tilde{\omega})$  space, namely  $(0, 1)^b \times (-K, K)^b$ , the following assumptions are verified for all  $r \geq 1$  and fixed  $\delta$  sufficiently small:

- (Hi)  $\text{supp } u^{(r)} \subseteq B(0, M^r)$  ( $\text{supp } u^{(0)} \subset B(0, M)$ )
- (Hii)  $\|\Delta u^{(r)}\| < \delta_r$ ,  $\|\partial \Delta u^{(r)}\| < \bar{\delta}_r$ , with  $\delta_{r+1} \ll \delta_r$  and  $\bar{\delta}_{r+1} \ll \bar{\delta}_r$ , where  $\partial$  refers to derivations in  $w$  or  $\tilde{\omega}$  and  $\|\cdot\| := \sup_{w, \tilde{\omega}} \|\cdot\|_{\ell^2(\mathbb{Z}^{b+d}) \times \ell^2(\mathbb{Z}^{b+d})}$ . (See the precise bounds at the conclusion of the proof.)
- (Hiii)  $|u^{(r)}(x)| < e^{-|x|^c}$  for some  $c \in (0, 1)$

Using (Hi-iii), an application of the implicit function theorem to the  $Q$ -equations:

$$\tilde{\omega}_k(w) = \frac{[(u + \bar{u})^{*p+1}](-e_k, j_k)}{2^{p+1}w_k \sqrt{j_k^2 + 1}} + \delta \frac{\hat{H}(-e_k, j_k)}{w_k \sqrt{j_k^2 + 1}}, \quad (6.1)$$

$k = 1, 2, \dots, b$ , with  $u = u^{(r)}$  yields

$$\tilde{\omega}_k^{(r)}(w) = \Omega_k(u^{(0)}(w)) + \delta^{1-\epsilon} \phi_k^{(r)}(w) \quad (6.2)$$



where the polynomials  $\Omega_k$ ,  $k = 1, 2, \dots, b$  are as in (3.1), homogeneous in  $w$  of degree  $p$ ,  $0 < \epsilon < 1/2$  in view of (4.18), and  $\|\partial\phi^{(r)}\| < C$ . We define  $\phi_0 = 0$  and denote the graph of  $\tilde{\omega}^{(r)}$  by  $\Phi_r$ . Moreover by (Hii),

$$|\tilde{\omega}^{(r)} - \tilde{\omega}^{(r-1)}| \lesssim \|u^{(r)} - u^{(r-1)}\| < \delta_r, \quad (6.3)$$

so that  $\Phi_{r-1}$  is an  $\delta_r$  approximation of  $\Phi_r$ . This can be seen as follows.

Consider the right side of (6.1) as a function of  $(w, \tilde{\omega})$  and rewrite (6.1) as

$$\tilde{f}_k(w, \tilde{\omega}) = 0,$$

for  $k = 1, \dots, b$ . Since

$$u^{(r')}(w, \tilde{\omega}) = u^{(0)}(w) + \sum_{i=1}^{r'} \Delta u^{(i)}(w, \tilde{\omega}),$$

$\tilde{f}_k$  may be written in the form:

$$\tilde{f}_k(w, \tilde{\omega}) := f_k(w, \tilde{\omega}) + P_k(u^{(0)}(w)),$$

where  $P_k$ ,  $k = 1, 2, \dots, b$ , are polynomials in  $w$ ,

$$P_k = \Omega_k + \delta \mathcal{O}(w^{p+1}).$$

Let  $X$  and  $Y$  be the partial derivative matrices:

$$X = \left[ \left[ \frac{\partial f_k}{\partial w_\ell} \right] \right] \text{ and } Y = \left[ \left[ \frac{\partial f_k}{\partial \tilde{\omega}_\ell} \right] \right], \quad k, \ell = 1, \dots, b.$$

We have

$$X = \mathcal{O}\left(\sum_{i=1}^{r'} \Delta u^{(i)}\right) + \mathcal{O}\left(\sum_{i=1}^{r'} \partial_w \Delta u^{(i)}\right)$$

and

$$Y = I + \mathcal{O}\left(\sum_{i=1}^{r'} \partial_{\tilde{\omega}}(\Delta u^{(i)})\right),$$

where the  $\mathcal{O}$  depends on  $u^{(0)}$  and (or)  $\partial_w u^{(0)}$ .

Using (Hii), the partial derivative matrix

$$\begin{aligned} \frac{\partial \tilde{\omega}}{\partial w} &:= \left[ \left[ \frac{\partial \tilde{\omega}_k}{\partial w_\ell} \right] \right], \quad k, \ell = 1, \dots, b \\ &= -Y^{-1}(X + \partial P / \partial w) \end{aligned}$$

is well-defined. The difference matrix satisfies

$$\left\| \frac{\partial \tilde{\omega}^{(r)}}{\partial w} - \frac{\partial \tilde{\omega}^{(r-1)}}{\partial w} \right\| \lesssim \|\partial_w \Delta u^{(r)}\| + \|\Delta u^{(r)}\|,$$

since

$$\Delta \tilde{\omega}^{(r)}(0) = \Delta u^{(r)}(0) = 0,$$

and  $|w| \leq 1$ , this proves (6.3).  $\square$

Below we continue with the assumptions on the *restricted* intervals in  $(w, \tilde{\omega})$  on  $(0, 1)^b \times (-K, K)^b$ , where approximate solutions could be constructed.

(Hiv) There is a collection  $\Lambda_r$  of intervals of size  $cM^{-r^C} \delta^\epsilon$ ,  $\epsilon \in (0, 1/2)$  such that

(a) On  $I \in \Lambda_r$ ,  $u^{(r)}(w, \tilde{\omega})$  is given by a rational function in  $(w, \tilde{\omega})$  of degree at most  $M^{Cr^3}$ , for some  $C > 1$

(b) For  $(w, \tilde{\omega}) \in \bigcup_{I \in \Lambda_r} I$ ,

$$\|F(u^{(r)})\| < \kappa_r, \quad \|\partial F(u^{(r)})\| < \bar{\kappa}_r \text{ with } \kappa_{r+1} \ll \kappa_r \text{ and } \bar{\kappa}_{r+1} \ll \bar{\kappa}_r$$

(c) Let  $N = M^r$ . For  $(w, \tilde{\omega}) \in \bigcup_{I \in \Lambda_r} I$ ,  $T = T(u^{(r-1)}) := F'(u^{(r-1)})$  satisfies

$$\|T_N^{-1}\| < M^{(r^C + |\log \delta|)},$$

$$|T_N^{-1}(x, y)| < e^{-|x-y|^c} \text{ for } |x-y| > Cr^C,$$

where  $T_N$  is  $T$  restricted to

$$[-N, N]^{b+d} \cup [-N, N]^{b+d} \sim [-N, N]^{b+d} \times \mathbb{Z}_2.$$

(d) Each  $I \in \Lambda_r$  is contained in an interval  $I' \in \Lambda_{r-1}$  and

$$\text{meas}_b(\Phi_r \cap (\bigcup_{I' \in \Lambda_{r-1}} I' \setminus \bigcup_{I \in \Lambda_r} I)) < \delta^{c\epsilon} [\exp \exp(\log(r+1))^{1/3}]^{-1}, \quad r \geq 2.$$

If  $\tilde{\omega} \in \Phi_r \cap I$ , then

$$\omega = \omega^{(0)} + \delta^p \tilde{\omega}$$

is Diophantine satisfying

$$\|n \cdot \omega\|_{\mathbb{T}} \geq \frac{\xi}{|n|^\gamma}, \quad \xi > 0, \quad \gamma > 6b,$$

for  $|n| \leq M^r$ , and

$$|P(\omega)| > \delta^p M^{-rc},$$

for all non-zero even quadratic polynomials

$$0 \neq P(\chi) \in \{[-CB^{2d}, CB^{2d}] \cap \mathbb{Z}\}[\chi_1, \chi_2, \dots, \chi_b],$$

where  $B = M^{r\rho}$  for some  $\rho > 0$ , after identification of  $\Phi_r \cap I$  with an interval in  $\mathbb{R}^b$ .

We remark that the approximate solutions  $u^{(r)}$  are defined, a priori, on  $\Lambda_r$ , but as *functions* they can be extended to  $(0, 1)^b \times (-K, K)^b$ , using a standard extension argument, cf. sect. 10, (10.33-10.37) in [B3], thus verifying (Hi-iii).

### *Proof of the Theorem*

The induction is as in Chaps. 18 and 19, the paragraphs from (19.76-19.81) of [B4]. It is also written in sect. 5.2 of [W1] with additional details. So below we only summarize a few key steps.

The first  $R$  ( $R = \mathcal{O}(|\log \delta|^{c'})$ ,  $0 < c' < 1$ ) steps of the induction are provided by the generalization of Proposition 4.3 to include all scales

$$N \in [M, e^{|\log \delta|^{s\bar{c}}}]$$

where  $s > 1$ , satisfies (5.33),  $0 < c' < s\bar{c} < 1$ . The iteration to subsequent scales uses Lemmas 5.1, 5.8 and 5.5. The following Lemmas play an essential role. They are proven in (19.76-19.81) in Chap. 19 of [B4] and sect. 5.2 of [W1].

Let  $u$  denote  $u^{(0)}, u^{(1)}, \dots$ . For all  $\bar{N}$ , let  $T_{\bar{N}} = T_{\bar{N}}(u)$  be the linearized operator evaluated at  $u$  and restricted to  $\{J + [-\bar{N}, \bar{N}]^{b+d}\} \times \{0, 1\}$ , where  $J \in \mathbb{Z}^d$ . (For simplicity the  $J$  subindex is omitted.) Define the operator  $T_{\bar{N}}(\theta)$  as before. Assume that (Hi-iv) hold at stage  $r$ . When  $\|J\|_\infty \leq \text{const. } \bar{N}$ , on the set of intervals  $\Lambda_r$  in (Hiv), there are moreover the following estimates.

**Lemma 6.1.** *There exist  $c, \sigma, \tau \in (0, 1)$  ( $c > \sigma > \tau$ ), as in Lemma 5.7, such that*

$$\begin{aligned} \|T_{\bar{N}}^{-1}(\theta)\| &< e^{\bar{N}^\sigma}, \\ |T_{\bar{N}}^{-1}(\theta)(x, y)| &< e^{-|x-y|^c}, \end{aligned}$$

for all  $x, y$  such that  $|x - y| > \bar{N}/10$ , away from a set  $B_{\bar{N}}(\theta)$  with

$$\text{meas } B_{\bar{N}}(\theta) < e^{-\bar{N}^\tau},$$

where  $u = u^{(r)}$ ,  $|\log \delta|^s \leq \bar{N} \leq r^C$ ,  $1 \leq 1/\sigma \leq s \leq 1/\tau$  as in (5.33),  $C > 1$ ,  $r \geq R$ .

There are related estimates on the set of intervals  $\Lambda_r$  when  $\|J\|_\infty > \text{const. } \bar{N}$ .

**Lemma 6.2.** *There exist  $c, \sigma, \kappa \in (0, 1)$  ( $c > \sigma > \kappa > \tau$ ), as in Lemma 5.5, such that*

$$\begin{aligned} \|T_{\bar{N}}^{-1}(\theta)\| &< e^{\bar{N}^\sigma}, \\ |T_{\bar{N}}^{-1}(\theta)(x, y)| &< e^{-|x-y|^c}, \end{aligned}$$

for all  $x, y$  such that  $|x - y| > \bar{N}/10$ , provided

$$\min_i |\theta - \theta_i(w, \tilde{\omega})| > e^{-\bar{N}^\kappa},$$

where  $\theta_i$  is a family of Lipschitz functions satisfying  $\|\theta_i\|_{\text{Lip}} \leq C\bar{N}$ ,  $i < \iota$  and  $\log \log \iota \sim \log \log \bar{N}$ ;  $u = u^{(r)}$ ,  $|\log \delta|^s \leq \bar{N} \leq r^C$ ,  $1 \leq 1/\sigma \leq s \leq 1/\tau$  as in (5.33),  $C > 1$ ,  $r \geq R$ .

Applying these lemmas and using (6.2, 6.3), for all  $r \geq 1$ , there are the bounds

$$\delta_r < \delta^{2-\epsilon} M^{-(\frac{4}{3})^r}, \bar{\delta}_r < \delta^{2-\epsilon} M^{-\frac{1}{2}(\frac{4}{3})^r}; \kappa_r < \delta^{p+3-2\epsilon} M^{-(\frac{4}{3})^{r+2}}, \bar{\kappa}_r < \delta^{p+3-2\epsilon} M^{-\frac{1}{2}(\frac{4}{3})^{r+2}},$$

$\epsilon \in (0, 1/2)$ , cf. the proofs of Lemmas 5.5 and 5.2 in [W1]. This iteratively solves the  $P$  and  $Q$ -equations.

From Proposition 3.1, for small  $\delta$ , the (closed) set  $\mathcal{B}'$  has a semi-algebraic description, in terms of one (non-zero) polynomial in  $w$  of degree  $b(p-1)$ , namely  $\det(\partial\Omega/\partial w)$ , where  $\Omega = \{\Omega_k\}_{k=1}^b$  and  $\Omega_k$  as in (3.1). (Recall that  $w = \delta^{-1}a$ .) The number of connected components of  $\mathcal{B}'$  is therefore at most  $C_b b^b (p-1)^b$ , cf. Theorem 1 in [Ba] or Theorem 9.3 in [B4]. The complement contains an open set of measure at least  $(1 - \epsilon'/2)$  in  $w$ . The proceeding construction proves the Theorem on a Cantor set of measure at least  $(1 - \tilde{\epsilon})$  for some  $0 < \epsilon' < \tilde{\epsilon} < 1$ . Renaming  $\tilde{\epsilon}$ ,  $\epsilon$  concludes the proof.  $\square$

## REFERENCES

- [Ba] S. Basu, *On bounding the Betti numbers and computing the Euler characteristics of semi-algebraic sets*, Disc. Comput. Geom. **22** (1) (1999), 1-18.
- [BeBo] M. Berti, Ph. Bolle, *Sobolev quasi-periodic solutions of multidimensional wave equations with a multiplicative potential*, Nonlinearity **25** (2012), 2579-2613.
- [B1] J. Bourgain, *Construction of periodic solutions of nonlinear wave equations in higher dimensions*, Geom. and Func. Anal. **5** (1995), 363-439.
- [B2] J. Bourgain, *Construction of approximative and almost periodic solutions of perturbed linear Schrödinger and wave equations*, Geom. and Func. Anal. **6** (1996), 629-639.
- [B3] J. Bourgain, *Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equations*, Ann. of Math. **148** (1998), 363-439.
- [B4] J. Bourgain, *Green's function estimates for lattice Schrödinger operators and applications*, Ann. of Math. Studies **158** (2005), Princeton University Press.

- [BGS] J. Bourgain, M. Goldstein, W. Schlag, *Anderson localization for Schrödinger operators on  $\mathbb{Z}^2$  with quasi-periodic potential*, Acta Math. **188** (2002), 41-86.
- [BW] J. Bourgain, W.-M. Wang, *Quasi-periodic solutions of nonlinear random Schrödinger equations*, J. Eur. Math. Soc. **10** (2008), 1-45.
- [BCN] H. Brezis, J.-M. Coron, L. Nirenberg, *Free vibrations for a nonlinear wave equation and a theorem of Rabinowitz*, Commun. Pure. Appl. Math. **33** (1980), 667-684.
- [CY] L. Chierchia, J. You, *KAM tori for 1D nonlinear wave equations with periodic boundary conditions*, Commun. Math. Phys. **211** (2000), 497-525.
- [G] V. Guillemin, *Lectures on spectral theory of elliptic operators*, Duke Math. J. **44** (1977), 485-517.
- [K] S. Kuksin, *Hamiltonian perturbation of infinite-dimensional linear systems with imaginary spectrum*, Funkts. Anal. i Prilozhen **21** (1987), 22-37.
- [P] J. Pöschel, *Quasi-periodic solutions for a nonlinear wave equation*, Comment. Math. Helvetici **71** (1996), 269-296.
- [Ra] P. H. Rabinowitz, *Free vibrations for a semi-linear wave equation*, Commun. Pure. Appl. Math. **31** (1978), 225-251.
- [R] K. Roth, *Rational approximations to algebraic numbers*, Mathematika **2** (1955), 1-20.
- [Ro] R. L. Roth, Classroom notes, *On extensions of  $\mathbb{Q}$  to square roots*, Amer. Math. Monthly **78** (1971), 392-393.
- [Schm] W. Schmidt, *Norm form equations*, Ann. of Math. **96** (1972), 526-551.
- [S1] I. Schur, *Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind, I*, J. Reine Angew. Math. **147** (1917), 205-232.
- [S2] I. Schur, *Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind, II*, J. Reine Angew. Math. **148** (1918), 122-145.
- [W1] W.-M. Wang, *Energy supercritical nonlinear Schrödinger equations: Quasi-periodic solutions*, Duke Math. J. **165** (2016), no. 6, 1129-1192.
- [W2] W.-M. Wang, *Quasi-periodic solutions for nonlinear wave equations*, C. R. Math. Acad. Sci. Paris **353** (7) (2015), 601-604.
- [Way] E. Wayne, *Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory*, Commun. Math. Phys. **127** (1990), 479-528.

CNRS AND DEPARTMENT OF MATHEMATICS, UNIVERSITÉ CERGY-PONTOISE, 95302 CERGY-PONTOISE CEDEX, FRANCE

*E-mail address:* wei-min.wang@math.cnrs.fr